Core-Selecting Auctions with Incomplete Information*

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Abstract

Core-selecting auctions were proposed recently as alternatives to the Vickrey-Clarke Groves (VCG) mechanism for environments with complementarities. Although the VCG mechanism is best motivated by its dominant-strategy property under incomplete information, the existing literature on core-selecting auctions is limited to only a complete information analysis. In this paper, we consider a simple incomplete-information model which allows us to do a full equilibrium analysis, including closed-form solutions for some distributions, for four different core-selecting auction formats suggested in the literature. Our model also admits correlations among bidders' values. We find that the revenues and efficiency from core-selecting auctions improve as correlations among bidders' values increase, while the revenues from the Vickrey auction worsen. Thus, there may be good reasons for policymakers to utilize a core-selecting auction rather than a VCG mechanism in realistic environments.

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1 Introduction

Core-selecting auctions have recently been proposed as alternatives to the Vickrey-Clarke-Groves (VCG) mechanism. In the VCG mechanism, the items are allocated so as to maximize values subject to the feasibility of the selected bids and each bidder is charged the opportunity cost of receiving the allocated items. While the mechanism has the attractive property that truth-telling is a dominant strategy — and truth-telling by all participants in the VCG mechanism implies efficient outcomes — there are several reasons to be wary of VCG in environments with complementarities. First, the VCG mechanism may generate *low revenues* (and, in environments with extreme complementarities, the revenues may equal zero). Second, VCG outcomes may be *non-monotonic* in the sense that increasing the number of bidders or increasing their valuations may reduce the seller's revenues. Third, the VCG mechanism may be especially vulnerable to unusual forms of *collusive behavior*, including collusion by losing bidders and shill bidding.

The simplest environment in which these issues can arise has just two items, "East" and "West" (which may be thought of as spectrum licenses for the Eastern half and Western half of a country), and three bidders. The "global" bidder views East and West as perfect complements, valuing the package {East, West} at 1, but obtaining no value from either item individually. Meanwhile, local bidder 1 values East at 1, but obtains no value from West; and local bidder 2 values West at 1, but obtains no value from East. Observe that the VCG mechanism[1] allocates East to local bidder 1 and West to local bidder 2, maximizing social surplus at 2. However, the mechanism charges a price of zero to each bidder.[2] The VCG outcome is non-monotonic in that, if each of the local bidders' values declined from 1 to ½, the seller's revenues would increase from 0 to 1. The explanation for low revenues, as well as for the opportunities present for loser collusion and shill bidding, is that the VCG outcome may lie outside the core;[3] with the data of this paragraph, a coalition of the seller and the global bidder can block the allocation at zero prices to the local bidders.[4]

Observe that the potential deficiencies of the VCG mechanism are likely to be empirically relevant. In the first place, much of the motivation for allowing package bidding in auctions arises from environments where there appear to be strong complementarities among items. Furthermore, in the area of telecommunications spectrum auctions, empirical work suggests that there exist substantial synergies among licenses covering different geographic areas.[5] Similarly, there is a growing interest in auctions with package bidding for financial assets, and this again occurs in environments where there are apparent complementarities among assets.

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[1] The VCG mechanism was developed in the work of Vickrey (1961), Clarke (1971) and Groves (1973). Throughout this paper, we will use the terms "VCG mechanism" and "Vickrey auction" interchangeably.

[2] Observe that the total surplus when local bidder 1 is absent equals 1, and so the incremental surplus created by local bidder 1 equals 1. Similarly, local bidder 2's incremental surplus also equals 1. In the VCG mechanism, each bidder is permitted to retain the entire incremental surplus that she creates, implying that the price paid by each local bidder is zero.

[3] The core is the subset of allocations in payoff space that are feasible and unblocked by any coalition.

[4] See Ausubel and Milgrom (2002).

[5] See, for example, Ausubel, Cramton, McAfee and McMillan (1997) and Fox and Bajari (2013).

As a result of this critique, researchers and auction practitioners recently began to explore a class of alternative mechanisms that have become known as *core-selecting auctions*. As in the VCG mechanism, buyers submit bids associated with various subsets of the set of all items, and the auctioneer determines the combination of bids which maximizes total value subject to feasibility. However, as seen two paragraphs above, applying the VCG payment rule in a complements environment may yield a profit allocation that lies outside the core. Instead, a core-selecting auction uses a different pricing rule — a rule always requiring the same or higher payments — which assures that the outcome is always in the core relative to the reported values.

Despite the very recent development of core-selecting auctions, they have already been selected for some important applications. At this writing, ten major spectrum auctions have been already conducted using a combinatorial clock auction (CCA) format: a two-stage auction procedure in which a simultaneous ascending clock phase is followed by a sealed-bid package auction. For the second stage, these auctions have utilized a core-selecting auction format with the nearest-Vickrey pricing rule. [6,7] The CCA format has also been selected for several upcoming spectrum auctions.[8]

However, to date, most studies of package bidding have been limited to complete-information analyses. This is not a particularly satisfying state of affairs, as much of the motivation for using VCG or other package-bidding mechanisms is that bidders possess incomplete and asymmetric information. At the same time, it is easy to understand why the shortcut of assuming complete information has typically been taken: analyses of auctions under incomplete information can be extremely intricate, except when truth-telling is an equilibrium. Moreover, the typical sort of environment motivating package bidding inherently includes asymmetries, as some bidders desire smaller sets of items and other bidders desire larger sets of items. Researchers have found that asymmetric auctions are particularly difficult to analyze.

A few contemporaneous papers have introduced explicit incomplete-information analyses of package bidding, but they are limited to considering independent valuations. Independence is an extremely confining assumption in an auction environment. In many of the most important applications of package bidding, such as spectrum auctions, we would expect there to be significant correlations among bidders' signals — and correlation among bidders' signals has been one of the important ingredients in the theory of auctions of single items. Moreover, a central message of auction theory and mechanism design is that, when correlations are present, particular choices of auction format may enhance the ability of the seller to extract revenues from bidders.[9]

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[6] The CCA has been used for spectrum auctions in the UK (three auctions: February, May 2008 and February 2013), the Netherlands (April 2010), Denmark (May 2010), Austria (two auctions: September 2010 and October 2013), Switzerland (March 2012), Ireland (October 2012) and Australia (May 2013).

[7] See Cramton (2013).

[8] The CCA has been adopted for the forthcoming Canada 700 MHz Auction, Canada 2500 MHz Auction and Slovakia Multiband Auction.

[9] See, for example, Milgrom and Weber (1982) and Crémer and McLean (1985).

The current paper seeks to advance the analysis of package bidding. We consider a very simple and stylized class of models in which one bidder values the items as perfect complements. We compare and contrast a variety of package bidding formats, including the core-selecting auctions in the literature, as well as the VCG mechanism.

Our model is an incomplete-information version of the auction environment with two items and three bidders that is described in the second paragraph of this Introduction. The global bidder obtains value u from winning *both* the Eastern and Western licenses, but gets zero value from having only East or West. Local bidder 1 values East at v_1, but obtains no value from West; while local bidder 2 values West at v_2, but obtains no value from East. The game is a standard Bayesian game in which each player knows the realization of her own value, but only the distribution from which her opponents' values were drawn. The players simultaneously and independently submit bids, where b_1 denotes the bid submitted by local bidder 1 for East, b_2 denotes the bid submitted by local bidder 2 for West, and B denotes the package bid submitted by the global bidder for {East, West}. The solution concept is Bayesian-Nash equilibrium.[10]

One of the novel aspects of our analysis is the family of distributions that we treat. The local bidders' values are perfectly correlated with probability γ and independently distributed with probability $1 - \gamma$. (Moreover, at the time that the local bidder selects her bid, she is unaware of whether the values are perfectly correlated or independent.) Thus, we consider a parameterized family of distributions that permits the correlation between local bidders' signals to be varied continuously from zero to one. Surprisingly, despite the private information and correlated signals, we are able to obtain explicit closed-form solutions for the core-selecting auction formats considered — for all $\gamma \in [0,1]$. And the possibility of positive correlation has a quite substantial impact on our comparison of the various package-bidding formats.

We know from Cremer-McLean (1985) and McAfee-Reny (1992) that the mechanism design approach will enable all informational rents to be extracted from the bidders with correlated types, enhancing both efficiency and revenues. However, the description of the optimal mechanism is sensitive to the data of the problem and the transfers to implement the mechanism will often swamp the underlying economic transaction. In particular, the bidders' payments in some states will exceed their values, creating major issues if bidders are risk averse or do not trust the seller. Consequently, we opt for analyzing specific sets of auction rules from the literature that are similar to rules in actual use and that are always ex post rational for bidders, rather than applying a mechanism design approach.

To demonstrate the various package-bidding mechanisms that we compare and contrast, suppose that the bids submitted by the respective bidders are $b_1 = 6$, $b_2 = 8$, and $B = 10$. In any of the mechanisms, the auctioneer first solves the winner determination problem of finding the allocation

[10]Further, the joint distribution of values will be symmetric with respect to the two local bidders, and we will limit attention to Bayesian-Nash equilibria that are symmetric with respect to the two local bidders. In addition, in all of the core-selecting auctions, the global bidder will have a weakly-dominant strategy, and we will then limit attention to Bayesian-Nash equilibria in which the global bidder plays her weakly-dominant strategy.

which maximizes value subject to the feasibility constraint. This bid data clearly results in local bidder 1 winning East and local bidder 2 winning West, as 6 + 8 = 14 > 10. The payments, p_1 and p_2, of local bidders 1 and 2, respectively, remain to be defined. The various mechanisms to be discussed in this paper will differ in their payment rules. Our analysis will consider the following package-bidding mechanisms:

> VICKREY-CLARKE-GROVES (VCG): Payments are determined such that each winner receives a payoff equal to the incremental surplus that she brings to the system. The incremental surplus of local bidder 1 equals 4, as surplus (evaluated using the bidders' bids) equals 14 if local bidder 1 is present, and 10 (the global bidder's value) if local bidder 1 is absent. Thus, $b_1 - p_1 = 4 \Rightarrow p_1 = 2$. Similarly, the incremental surplus of local bidder 2 equals 4, so $b_2 - p_2 = 4 \Rightarrow p_2 = 4.$ Thus, the VCG payments are $(p_1, p_2) = (2,4)$. However, this outcome is not in the core, as the seller and the global bidder form a *blocking coalition*: together, they can realize surplus of 10 (the global bidder's value), while in the VCG outcome, the seller receives payoff of $6 = p_1 + p_2$ and the global bidder receives payoff of 0.
>
> NEAREST-VICKREY: Payments are determined such that the profit allocation is the bidder-optimal core allocation that minimizes the Euclidean distance from the VCG outcome. In order to avoid the presence of any blocking coalitions, the payments, p_1 and p_2, must sum to at least 10; and in a bidder-optimal core allocation, the payments must sum to exactly 10. The payments that satisfy this restriction and minimize the distance from the VCG payments of (2,4) are $(p_1, p_2) = (4,6)$.
>
> PROXY AUCTION: Payments are determined that reflect the outcome of "proxy agents" competing in a simultaneous ascending auction with package bidding and arbitrarily small bid increments, ε. The bids b_1, b_2 and B are reinterpreted as limit prices that the bidders have given their respective proxy agents. Each proxy agent must bid in the "virtual auction" whenever it is not a provisionally-winning bidder. In the initial round, all three proxy agents submit bids of ε, making the two local bidders provisional winners. In round two, the proxy agent for the global bidder raises its bid to 3ε, making the global bidder the provisional winner; in round three, the proxy agents for each of the local bidders raise their bids to 2ε, making the two local bidders provisional winners; and the process repeats until the proxy agent for the global bidder drops out of the auction at a price of essentially $B = 10$. Thus, the local bidders win the virtual auction at prices of essentially $(p_1, p_2) = (5,5)$, an alternative bidder-optimal core outcome.
>
> PROPORTIONAL PRICING: Payments are determined such that the bids are scaled down, proportionally, until the bidder-optimal frontier of the core is reached. In the above example, the bids of the local bidders sum to 14, and so they can each be scaled down by a factor of 5/7 in order to sum to 10. Thus, the payments are $(p_1, p_2) = (30/7, 40/7)$.
>
> NEAREST-BID: Payments are determined such that the profit allocation is the bidder-optimal core allocation that minimizes the Euclidean distance from the vector of winning bids. In the

above example, the bidder-optimal core payments that minimize the distance from the winning bids of (6, 8) are $(p_1, p_2) = (4,6)$, coinciding with the nearest-Vickrey outcome in this example, but not in general.

FIRST-PRICE PACKAGE: Payments simply correspond to the amounts of the winning bids. That is, the auctioneer first solves the winner determination problem of finding the allocation which maximizes revenues subject to the feasibility constraint; and the winning bidders' required payments are simply the amounts of their winning bids. If the same bids were submitted as in the above example, then the payments would be $(p_1, p_2) = (b_1, b_2) = (6,8)$. Obviously, since this is a "first price" rather than a "second price" auction format, it should be expected that bids would be substantially different from those in the other core-selecting auctions.

Each of these package-bidding pricing rules, as applied to the bid data $b_1 = 6$, $b_2 = 8$, and $B = 10$, is illustrated in Figure 1. Observe that the set of prices associated with core allocations is the shaded triangle of this figure, while the set of bidder-optimal core prices is the hypotenuse of this triangle.

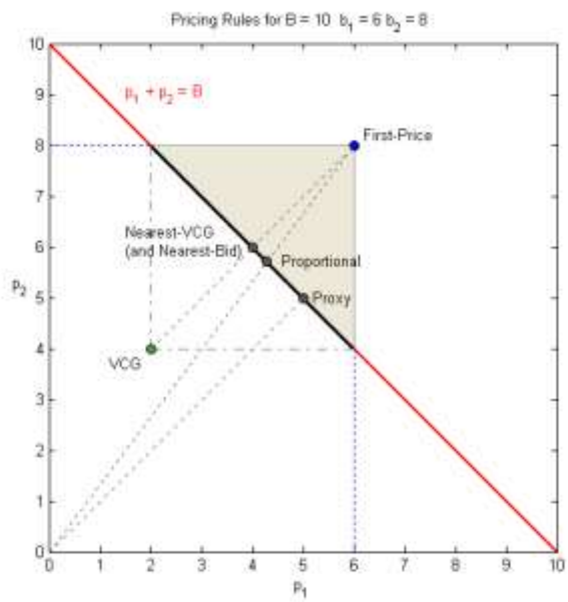

Figure 1. Package-Bidding Pricing Rules (as applied to example bid data)

For all but the last of the core-selecting auctions listed above, if the marginal distribution of each bidder's value is the uniform distribution, then we are able to derive explicit closed-form solutions for equilibria, for all $\gamma \in [0,1]$. However, for the first-price package auction, our analysis does not yield a solution—the explanation is that our approach relies on the global bidder being charged a "second price", and therefore having truthful bidding as a weakly-dominant strategy. Nevertheless, we believe that the comparison is of considerable interest. To compare the performance of the first-price package auction with the other core-selecting auctions, we report the revenues and efficiency as computed using a numerical technique for approximating equilibria that is introduced in Baranov (2013).

It is well understood from the literature that the model under consideration presents local bidders with a severe free-rider (or "threshold") problem.[11] The effects of various pricing rules on the free-rider problem can be very different. While it might also be desirable to consider them in richer environments, our analysis provides a very clear intuitive understanding of the free-rider incentives under each pricing rule. The model considered is the simplest possible abstraction in which this issue is fully present; it is also sufficiently elegant that exact closed-form solutions are possible under second-price rules. As such, we believe that the simple model is very well suited for studying this question.

The VCG mechanism was introduced in the classic theory of auctions and public choice. William Vickrey (1961) treated auctions with multiple units of a homogeneous product, while Edward Clarke (1971) and Theodore Groves (1973) treated public choice problems. The Clarke-Groves treatment subsumed the environment that Vickrey studied as well as auctions of multiple heterogeneous objects. We use the terminology "VCG mechanism" and "Vickrey auction" interchangeably.

The study of second-price-like auction mechanisms generating core allocations originated with Ausubel and Milgrom (2002, 2006). They described an "ascending proxy auction" mechanism which was proven to yield core allocations with respect to bidders' reports and whose complete-information equilibrium allocations (subject to a refinement) coincided with the set of bidder-optimal core allocations. A closely-related auction procedure was developed independently by Parkes and Ungar (2000) and Parkes (2001). Stated somewhat imprecisely, the outcome of the ascending proxy auction coincides with that of the VCG mechanism if and only if all bidders have substitutes preferences.[12] Ausubel, Cramton and Milgrom (2006) then proposed a two-stage auction procedure comprising a (multi-round) ascending-clock auction followed by a single proxy auction round. This became the basis for the "package clock" (or "combinatorial clock") auction design adopted by the UK and other governments for spectrum auctions (Cramton, 2013). Hoffman, Menon, van den Heever and Wilson (2006) introduced acceleration techniques for computing the proxy auction.

Day and Raghavan (2007) and Day and Milgrom (2008) independently introduced the notion of core-selecting auctions. Each pair of authors proposed a generalization of the proxy auction where the "virtual" auctions of the proxy are superseded by a direct consideration of core allocations relative to bidders' reports, and each demonstrated the incentive advantages of selecting the bidder-optimal core allocation. Day and Raghavan introduced a core constraint generation algorithm which is an especially effective method for calculating bidder-optimal core allocations and, in particular, advocated the bidder-optimal core allocation that minimizes the maximum deviation from the VCG

[11] See Krishna and Rosenthal (1996).

[12] More precisely, if all bidders have substitutes preferences, then the VCG allocation is in the core. Conversely, if there are at least four bidders, if the set of each bidder's possible valuations includes the additive valuation functions and if at least one bidder's possible valuations includes non-substitutes preferences, then there exists a profile of bidder valuations such that the VCG allocation is not in the core. (See Ausubel and Milgrom (2002).)

payments. Day and Milgrom proved important economic properties, including that an efficient direct mechanism is immune from shill bidding if and only if it is a core-selecting auction. Meanwhile, Day and Cramton (2012) proposed the nearest-Vickrey pricing rule and demonstrated how to compute it efficiently.

Three other recent papers have begun to explore the comparison among core-selecting auctions. Erdil and Klemperer (2010) define a class of payment rules referred to as "reference rules" — the proxy auction's payment rule is one example, while the nearest-Vickrey rule is not — and they argue that reference rules reduce the marginal incentive to deviate as compared to other payment rules. While their paper does not explicitly contain incomplete-information analysis, their conclusions foreshadow the results of the current paper. Goeree and Lien (2013) consider the incomplete-information game with a global bidder and two local bidders whose valuations are independent and uniformly-distributed. Simultaneously and independently from the current paper, they solve for the Bayesian-Nash equilibrium of the nearest-Vickrey pricing rule for independent uniform distributions and they find that the VCG mechanism dominates it in expected revenues as well as efficiency. Sano (2011) considers the incomplete-information game with a global bidder and two local bidders whose valuations are independent and uniformly-distributed. Simultaneously and independently from the current paper, he solves for the Bayesian-Nash equilibrium of the proxy auction under independence, finding that high-value local bidders submit almost their true values, while low-value local bidders shade considerably.

In the current paper, we too analyze incomplete-information games with a global bidder and two local bidders. We formulate the game and solve for equilibria, allowing independence ($\gamma = 0$) or correlation ($\gamma > 0$) between the local bidders' values. We consider four different core-selecting auctions — the nearest-Vickrey, the proxy, the proportional and the nearest-bid pricing rule — and for each $\gamma \in [0, 1]$, we are able to obtain explicit closed-form solutions under certain assumptions on the distributions.[13] For the case where the marginal distributions are uniform, we obtain Figure 2, which summarizes the expected seller revenues and efficiency in the equilibrium:

[13] Only three solutions are required, as the equilibrium for the proportional pricing rule coincides with the equilibrium for the nearest-Vickrey rule in the model we consider.

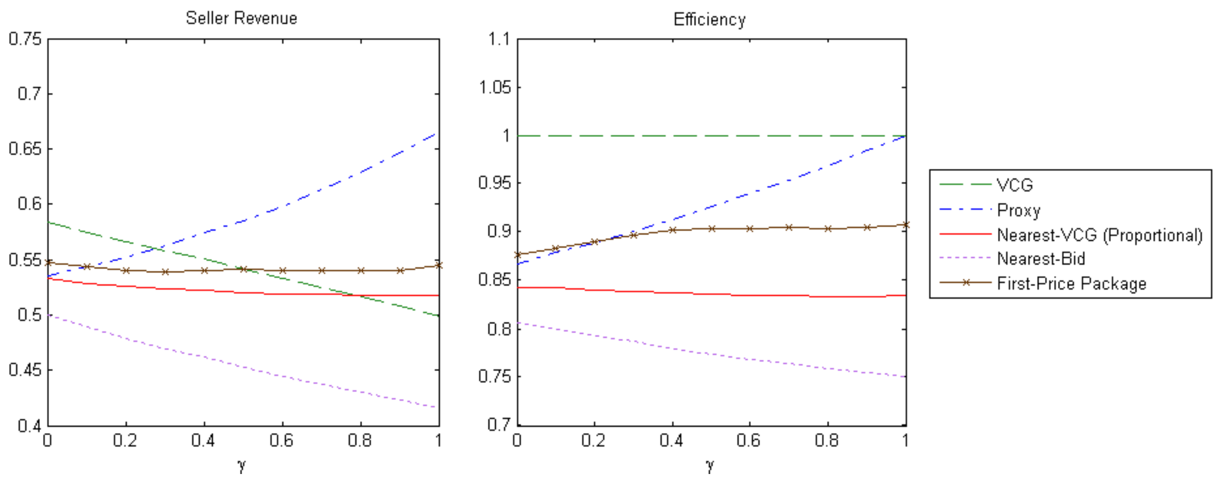

Figure 2. Seller Revenue and Efficiency for $\alpha = 1$ and all $\gamma \in [0,1]$

Counter to Goeree and Lien, we find that the choice between a core-selecting auction or the VCG mechanism is sensitive to the information structure. As shown in Figure 2, the relative performance of the alternative mechanisms changes substantially as the correlation γ increases from zero to 1. When $\gamma = 0$, the VCG mechanism raises 8.9% higher expected revenues than the proxy auction, and it achieves 9.6% higher revenue than the nearest-Vickrey rule, despite achieving greater efficiency. However, at the opposite extreme, when $\gamma = 1$, the proxy auction attains 33.3% higher expected revenues than the VCG mechanism while also realizing full efficiency. Clearly, the choice of whether to use the VCG mechanism or a core-selecting auction depends on the likely informational environment.

We also consider the effect of varying the uniform distribution on the local bidders' values. While the distribution $F(v) = v^{\alpha}$ does not generally admit a closed-form solution, it does when $\alpha = 2$. Stable numerical simulations can be found for other α — for symmetry with $\alpha = 2$, we also consider $\alpha = ½$. In this formulation, α controls the relative frequency of local bidders' winnings under full efficiency. For example, when $\alpha = 1$ (uniform distribution) local bidders are expected to win with probability ½ while they only expected to win with probability 1/3 when $\alpha = ½$. We find that the comparison among the VCG mechanism and the various core-selecting auctions changes in α. In particular, the case of $\alpha = 2$ reverses the revenue ranking of the proxy auction and the VCG mechanism, while the case of $\alpha = ½$ enhances the revenue advantage of VCG emphasized by Goeree and Lien.

This paper proceeds as follows. In Section 2, we present the model, including the family of distributions that allow partial correlation, and we detail the package-bidding mechanisms to be considered. In Section 3, we introduce two important properties for core-selecting auctions and we establish Lemma 4, which provides local optimality conditions for local bidders in any mechanism satisfying these properties. We solve for explicit closed-form solutions for the various mechanisms under consideration, for all correlation parameters, in Section 4. In Section 5, we discuss extensions to the basic model, and we consider one policy application in Section 6. Section 7 concludes. Most proofs are relegated to Appendix A, and the solutions for different values of parameters γ and α are summarized in Appendix B.

2 The Model

Two items are offered for sale. There are two local bidders, 1 and 2, who are interested in only one item and receive no extra utility from acquiring the second item. Their values are denoted v_1 and v_2, respectively. There is one global bidder who wants to acquire both items and obtains no utility from owning just one item. Her value for the pair of items is denoted u. The bidders are risk neutral and have quasilinear utilities: the payoff of local bidder i, if she wins one unit at price p_i, is $v_i - p_i$; and the payoff of the global bidder, if she wins both units at a total price of p, is $u - p$.

The value, u, of the global bidder is independently drawn from the distribution on $[0,2]$ described by a cumulative distribution function $G(u)$ with atomless probability density function $g(u)$. With probability γ, the values, v_i ($i = 1, 2$), of local bidders are perfectly correlated and drawn from a distribution on the interval $[0,1]$, defined by a cumulative distribution function $F(v)$ with atomless density $f(v)$. With probability $1 - \gamma$, the values of the local bidders are independently drawn from the same distribution $F(v)$.

It is economically reasonable to expect correlations in values among all three bidders in the model. For example, in a spectrum auction, the three firms might plan to deploy similar telecommunications technologies. To simplify the model, any global-local correlations are abstracted away; including them would merely introduce correlations between competing bidders that are well understood in the literature. However, we preserve the local-local correlations, as they are essential to our analysis. These are value correlations between two bidders who need to cooperate (to outbid the global bidder), and the exact structure of the pricing rule turns out to affect their ability to cooperate.

Parameter γ controls the amount of correlation between the local bidders' values. The local bidders' value model is summarized by the conditional cumulative distribution function of the local bidder i given her value v_i:

$$F_L(v_j \mid v_i = x) = \begin{cases} (1-\gamma)F(v_j), & \text{if } v_j < x \\ (1-\gamma)F(v_j) + \gamma, & \text{if } v_j \geq x \end{cases} \quad \text{for } i \neq j \ .$$

It is interesting to observe that v_i and v_j are *not* affiliated random variables for any $\gamma > 0$. [14] Nevertheless, for $y \geq x$, $F_L(\bullet \mid y)$ (first-order) stochastically dominates $F_L(\bullet \mid x)$. The failure of affiliation would prevent some of the results in the theory of single-item auctions from going through. (However, as noted above, the structure of winning is different in the current package-bidding context: local bidder 1's bid need not exceed the bid of local bidder 2; rather the sum of the bids of

[14]Consider $x > y > z$ and let $\Xi(\cdot,\cdot)$ denote the joint probability of v_i and v_j. Then $(y, y) \vee (x,z) = (x,y)$ and $(y, y) \wedge (x,z) = (y,z)$, but $\Xi(x,y)\,\Xi(y,z) < \Xi(y,y)\,\Xi(x,z)$, contradicting the affiliation inequality.

local bidders 1 and 2 needs to exceed the bid of the global bidder.) In any case, the correlation model of this paper cannot be used in standard single-item auctions.[15]

Our model handles both homogeneous and heterogeneous environments. In the former interpretation, local bidder *i* derives positive utility v_i from winning either item.[16] In the latter interpretation, there are two heterogeneous items, East and West; local bidder 1 obtains positive utility only from East and local bidder 2 obtains positive utility only from West.[17] Our equilibrium solutions are fully consistent with either interpretation.

All of the auction mechanisms that we analyze in this paper, other than the VCG mechanism, satisfy the following definition:

Definition 1. A *core-selecting auction* is a mapping from bids to allocations and payments such that the payoffs resulting from every bid profile are elements of the core.[18]

Without loss of generality we limit our attention to the restricted auctions in which each bidder is allowed to submit only one bid.[19] While impractical in a general environment, this limitation does not affect efficiency in any way because of the single-minded nature of the bidders' preferences in the model we consider here. For example, the global bidder has value for a package of two items and her bid B is interpreted as a package bid for two items. Each local bidder i is interested only in one item and her bid b_i expresses her willingness to pay up to b_i for the one item.

All auctions considered in the paper proceed in the following manner. First, all bidders submit their bids to the auctioneer who then chooses an allocation which maximizes total welfare with respect to the bids. In our simple model, only two outcomes are possible. If the package bid of the global bidder is greater than the sum of the local bids, i.e. $B > b_1 + b_2$, the global bidder wins the auction and receives both items. The local bidders win the auction and receive one item each whenever the sum of their bids is higher than the package bid of the global bidder, i.e. $B < b_1 + b_2$ .Ties are resolved using a fair randomizing device. The payment each winner is required to make depends on a specific pricing rule.

We consider the VCG mechanism and several core-selecting pricing rules. Denote V_1 and V_2, the VCG payments of local bidders in case of winning, i.e. $V_1 = \max\{0, B - b_2\}$, $V_2 = \max\{0, B - b_1\}$. Additionally, we use $p(b_1, b_2, B)$ to denote a payment vector associated with the corresponding bids b_1, b_2 by local bidders and a bid B by the global bidder.

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[15] For example, consider a symmetric first-price auction with two bidders whose values are correlated in the same way. If one bidder knows that the other bidder has the same value with a positive probability, her best response fails to exist.

[16] Then the global bidder exhibits classic increasing returns to scale.

[17] Then the global bidder is intending to implement a technology which (for technical or marketing reasons) is only economical if deployed on a nationwide basis.

[18] This definition is taken from Day and Milgrom (2008).

[19] See Beck and Ott (2013) for subsequent analysis of core-selecting auctions when bidders are allowed to place bids on unwanted items.

Without loss of generality, we will assume that $b_1 \geq b_2$.

(1) VCG Mechanism (Benchmark Rule)

This is a well-known pricing rule which is motivated by its dominant strategy property. Under this rule, the payment of the particular bidder does not depend upon her bid and only affects the allocation:

$$p(b_1, b_2, B) = \begin{cases} (V_1, V_2, 0) & \text{if } \ \mathrm{B} < b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if } \ \mathrm{B} > b_1 + b_2 \end{cases}.$$

(2) Proxy Rule

The *ascending proxy auction* was suggested by Ausubel and Milgrom (2002). Given our simple model, it can be summarized using the following formula:

$$p(b_1, b_2, B) = \begin{cases} (\frac{1}{2}B, \frac{1}{2}B, 0) & \text{if } \quad B \leq 2b_2 \\ (B - b_2, b_2, 0) & \text{if } \quad 2b_2 < B < b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if } \quad B > b_1 + b_2 \end{cases},$$

or, equivalently,

$$p(b_1, b_2, B) = \begin{cases} (\frac{1}{2}B, \frac{1}{2}B, 0) & \text{if } \quad B \leq 2b_2 \\ (V_1, B - V_1, 0) & \text{if } \quad 2b_2 < B < b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if } \quad B > b_1 + b_2 \end{cases}.$$

(3) Nearest-VCG Rule

The nearest-VCG pricing rule was introduced by Day and Cramton (2009), superseding Day and Raghavan's (2007) suggestion of minimizing the maximum deviation from the VCG payments. The central idea of this rule is to select the bidder-optimal core allocation that minimizes the Euclidean distance to the VCG point:

$$p(b_1, b_2, B) = \begin{cases} (V_1 + \Delta, V_2 + \Delta, 0) & \text{if } \quad \mathrm{B} < b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if } \quad \mathrm{B} > b_1 + b_2 \end{cases}$$

where $\Delta = \dfrac{B - V_1 - V_2}{2}$.

(4) Proportional Rule

This is a natural rule to consider in this environment. Whenever the local side wins the auction, they split the amount they are required to pay proportionally to their bids:

$$p(b_1, b_2, B) = \begin{cases} \left(\dfrac{b_1}{b_1 + b_2} B, \dfrac{b_2}{b_1 + b_2} B, 0 \right) & \text{if } \ B < b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if } \ B > b_1 + b_2 \end{cases}.$$

(5) Nearest-Bid Rule

The “nearest-bid” description corresponds to the point in a minimum-revenue core which is the closest to the winners’ bids. This rule can be motivated by a simple description of the payment procedure. In case of winning each local bidder pays her bid and then gets a refund. The amount of the refund is just half of the “money left on the table”, i.e. $b_1 + b_2 - B$. As with Proxy Rule, if bids are too different, the amount of refund might be higher than the smallest of the locals’ bids. Since payments cannot be negative, the local bidder i with the small bid $\left(b_i \ll b_j\right)$ is reimbursed completely while the local bidder j pays the global bidder’s bid alone. This rule is intuitive and easy to explain to the bidders:

$$p(b_1, b_2, B) = \begin{cases} (B, 0, 0) & \text{if} \quad B \le b_1 - b_2 \\ (b_1 - \Delta, b_2 - \Delta, 0) & \text{if} \quad b_1 - b_2 < B < b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if} \quad B > b_1 + b_2 \end{cases},$$

where $\Delta = \dfrac{b_1 + b_2 - B}{2}$.

3 Initial Analysis

Definition 2. A bid b by bidder i is *pivotal* if, for any $\varepsilon > 0$, a bid $b + \varepsilon$ yields bidder i a non-empty set of items, while a bid of $b - \varepsilon$ yields bidder i the empty set.

Note that, in auctions with pricing rules (1)–(5), any bid $(b_1, b_2 \text{ } or \text{ } B)$ is pivotal if and only if $b_1 + b_2 = B$.

Definition 3. An auction satisfies the *pivotal pricing property* with respect to a given bidder if, whenever the bidder's bid is pivotal, the price that she pays (if she wins) equals her bid.

The pivotal pricing property is very natural and is satisfied for the most of the reasonable auction formats. Consider standard single-item auction with at least three bidders. First-price and all-pay auctions necessarily satisfy the pivotal pricing property since the winner always pays her bid. In a second-price auction a winning bid is pivotal only if top two bids are equal to each other in which case the winner pays her bid precisely. However, some auctions do not satisfy this property. For example, in a third-price auction a winner with pivotal bid in general pays less than her bid.

Lemma 0. The VCG mechanism satisfies the pivotal pricing property with respect to all bidders.

Proof: If bidder i’s bid, b_i, is pivotal, then the incremental surplus contributed by bidder i is zero. By the specification of the VCG mechanism, bidder i’s payoff in the mechanism equals zero. Consequently, bidder i pays a price of b_i.

Lemma 1. Every core-selecting auction satisfies the pivotal pricing property with respect to all bidders.

Proof: Let p_i denote the price paid by bidder i when her bid, b_i, is pivotal, and let S_i denote the set of winning bidders if bidder i had instead submitted a bid of $b_i - \varepsilon$. By the definition of a pivotal bid, $i \notin S_i$. Suppose that $p_i < b_i$. Then the allocation can be blocked by the coalition comprising the seller and set S_i. Suppose instead that $p_i > b_i$. Then the allocation can be blocked by the coalition comprising bidder i alone. We conclude that $p_i = b_i$.

Definition 4. An auction satisfies the *continuous pricing property* with respect to a given bidder if, as long as the bidder's allocation stays the same, the price that she pays is a continuous function of all bids.

Similar to the pivotal pricing property, the continuous pricing property is satisfied for the majority of reasonable auction formats. It simply means that the bidder's payment does not change discontinuously without a corresponding change in the bidder's allocation. The following lemma verifies that all pricing rules considered in the paper satisfy this property.

Lemma 2. Auctions with pricing rules (1) – (5) satisfy the continuous pricing property with respect to all bidders.

Proof: All pricing rules (1) – (5) can be verified to satisfy the continuous pricing property.

Remark: It is straightforward to construct pricing rules that violate the continuous pricing property. Consider the following "Reversed Proxy" Rule:

$$p(b_1, b_2, B) = \begin{cases} (\frac{1}{2}B, \frac{1}{2}B, 0) & \text{if} \quad B < 2b_2 \\ (B - V_2, V_2, 0) & \text{if} \quad 2b_2 \le B < b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if} \quad B > b_1 + b_2 \end{cases} .$$

This rule is defined identically to the Proxy Rule except when local bidders win but one of them bids less than half of the global bidder's bid. In such situations, the Reverse Proxy has the core burden borne by the local bidder with the higher bid, whereas the Proxy had the burden borne by the local bidder with the lower bid.

Suppose the following profile of bids: $b_1 = 5$, $b_2 = 8$, and $B = 10$. With these bids, local bidders win and pay 5 each. However, if local bidder 1 had reduced her bid to $5 - \varepsilon$, she would only pay 2 and the other local bidder would pay 8, contradicting the continuous pricing property.

Lemma 3. The global bidder has a weakly dominant strategy to bid her value in auctions with pricing rules (1) – (5).

Proof: For each of these pricing rules, the global bidder wins if and only if her package bid, B, satisfies $B \geq b_1 + b_2$, and her payment is then $b_1 + b_2$. Consequently, the exact same argument holds as in the standard second-price auction for a single item.

In what follows, we assume that the global bidder bids according to her weakly dominant strategy, i.e. $B(u) = u$.

With a slight abuse of notation let $\beta(.)$ denote the symmetric equilibrium bid function of the local bidders for all pricing rules. Additionally, denote $\Phi_i(b_i, v_i)$ and $\varphi_i(b_i, v_i)$ the probability of winning and marginal probability of winning for a local bidder i who submits a bid b_i assuming all other bidders follow their equilibrium strategies, i.e.:

$$
\begin{aligned}
\Phi_i(b_i, v_i) &= \Pr(b_i + b_j > B) = \\
&= \gamma \int\limits_{b_i + \beta(v_i) > B(u)} g(u)\,du \quad + \quad (1-\gamma) \iint\limits_{b_i + \beta(v_j) > B(u)} f(v_j) g(u)\,dv_j du = \\
&= \gamma G(b_i + \beta(v_i)) + (1-\gamma) \int\limits_{v_j} f(v_j) G(b_i + \beta(v_j))\,dv_j \\
\varphi_i(b_i, v_i) &= \frac{\partial \Phi_i(b_i, v_i)}{\partial b_i} \\
&= \gamma g(b_i + \beta(v_i)) + (1-\gamma) \int\limits_{v_j} f(v_j) g(b_i + \beta(v_j))\,dv_j \quad .
\end{aligned}
$$

Let $P_i(b_i, v_i)$ and $MP_i(b_i, v_i)$ denote the expected payment and the expected marginal payment, respectively, for a local bidder who submits a bid b_i assuming all other bidders follow their equilibrium strategies, i.e.:

$$
\begin{aligned}
P_i(b_i, v_i) &= Ep_i(b_i, b_j, B) = \\
&= \gamma \int\limits_u p_i(b_i, \beta(v_i), u) g(u)\,du + (1-\gamma) \iint\limits_{u, v_j} p_i(b_i, \beta(v_j), u) f(v_j) g(u)\,dv_j du \\
MP_i(b_i, v_i) &= Ep_i'(b_i, b_j, B) = \\
&= \gamma \int\limits_u p_i'(b_i, \beta(v_i), u) g(u)\,du + (1-\gamma) \iint\limits_{u, v_j} p_i'(b_i, \beta(v_j), u) f(v_j) g(u)\,dv_j du \quad .
\end{aligned}
$$

Lemma 4. For an auction satisfying *the pivotal pricing property* and *the continuous pricing property*, the optimality conditions for a local bidder i are given by:

$$
\begin{aligned}
&(v_i - b_i)\,\varphi_i(b_i, v_i) \leq MP_i(b_i, v_i) \quad b_i \geq 0 \\
&b_i \left[(v_i - b_i)\,\varphi_i(b_i, v_i) - MP_i(b_i, v_i) \right] = 0 \quad .
\end{aligned}
$$

Proof: See Appendix A.

Lemma 4 just simplifies the Karush-Kuhn-Tucker conditions for the local bidders' profit maximization problem taking into account both the pivotal pricing and continuous pricing properties. Intuitively, an infinitely small increase in a bid affects costs by potentially increasing expected payment in non-pivotal states and adding a new payment in the pivotal state (when the increase results in a pivotal bid, or the state in which bidder wins only because she increased her bid by a small amount). The latter payment equals to the player's bid according to the pivotal pricing property.

In case $\varphi_i(b_i, v_i) > 0$, the optimality conditions in Lemma 4 can be rewritten as:

$$b_i = \max\left(0, v_i - \frac{MP_i(b_i, v_i)}{\varphi_i(b_i, v_i)}\right).$$

Note that a local bidder shades her bid when the expected marginal payment is positive. We formalize this general functional form of the equilibrium local's bidder bid function in a Corollary 1.

Corollary 1. The general functional form of the locals' equilibrium bid function is

$$\beta(v) = \begin{cases} 0 & v \le d(\gamma) \\ c(v) & v > d(\gamma) \end{cases}$$

where

- $\begin{cases} d(\gamma) = d & \gamma < 1 \\ d(1) = 0 & \gamma = 1 \end{cases} \quad d : d \ge 0 \quad \text{such that} \quad \varphi(0, d)\, d = MP(0, d)$
- $c(v)$ is strictly increasing on $[d(\gamma), 1]$

The equilibrium bid function potentially has a flat segment in the beginning. Intuitively, the local bidder might find it optimal to free-ride on the other local bidder because the probability of winning is strictly greater than zero for a local bidder with a zero bid.

Proposition 0. The equilibrium bid function of local bidders under the VCG pricing rule is given by $\beta(v) = v$.

Proof: Well known.

4 Main Results

This section contains our main results. In order to derive equilibrium bids explicitly, we assume uniform distributions for all values. Namely, $f(.)$ is a uniform density on $[0,1]$ and $g(.)$ is a uniform density on $[0,2]$. Under this assumption, there is symmetry between global and local sides of the market because under full efficiency the global and local sides are expected to win equally often.

We start by considering the Proxy Rule.

Proposition 1. The equilibrium bid function of local bidders (in symmetric Bayesian-Nash equilibria) under the Proxy Rule is given by:

$$\beta(v)=\begin{cases} 0 & v \le d(\gamma) \\ 1+\dfrac{\ln\left(\gamma+(1-\gamma)v\right)}{1-\gamma} & v > d(\gamma) \end{cases} \quad \text{if} \quad \gamma<1 \qquad \text{and} \qquad \beta(v)=v \quad \text{if} \quad \gamma=1,$$

[20]

where $d(\gamma)=\dfrac{e^{-(1-\gamma)}-\gamma}{1-\gamma}>0 \quad \forall \gamma<1.$

Proof: See Appendix A.

Figure 3 (left panel) provides examples of equilibrium bid functions for the proxy rule. In equilibrium, local bidders with low values prefer to bid zero in an attempt to free-ride. Moreover, the size of the zero-bid interval is magnified by the proxy rule itself because a local bidder with a sufficiently small bid would be required to pay her bid whenever the locals win with the other local bidder paying the rest. To put it differently, a local bidder with a small bid in the proxy auction has shading incentives which are similar to that of the first-price package auction. In a sharp contrast, a high-type local bidder bids almost truthfully because she expects to be the highest bidder from the local side in which case she is very likely to pay her Vickrey payment.

With the increase in correlation, the zero-bid interval vanishes since a low-type local bidder no longer expects a sufficiently high bid from the other local bidder. Instead she expects a comparably low bid which makes her reluctant to shade. At the extreme case of perfect correlation, both local bidders bids truthfully in a symmetric equilibrium. The case of perfect correlation is very interesting since the proxy rule is able to achieve the first-best by combining equilibrium truthful-bidding property with the core property.

However, the proxy rule model with perfect correlation also has a multiplicity of other, asymmetric equilibria where revenue and efficiency performance is undermined. Specifically, one of the asymmetric equilibria results in truthful bidding by one of the local bidders and bidding zero by the other local bidder.

Proposition 2. The equilibrium bid function of local bidders under the Nearest-Vickrey Rule is given by:

[20] A symmetric Bayesian-Nash equilibrium for the Proxy Rule with local bidders having independent values, i.e. $\gamma = 0$, was derived independently in Sano (2011).

$$\beta(v)=\begin{cases}0 & v\le d(\gamma)\\ k(\gamma)v-d(\gamma) & v>d(\gamma)\end{cases}\quad \text{if}\quad \gamma<1^{21}\qquad \text{and}\qquad \beta(v)=\frac{2}{3}v\quad \text{if}\quad \gamma=1,$$

$$\text{where}\qquad k(\gamma)=\frac{2}{2+\gamma}\qquad d(\gamma)=\frac{2\left(\sqrt{-\gamma^2+2\gamma+8}-3\right)}{\gamma^2+\gamma-2}>0\quad \forall\gamma<1.$$

Proof: See Appendix A.

The equilibrium bid functions for the nearest-Vickrey rule are shown in Figure 3 (central panel). The size of the zero-bid interval is smaller when comparing to the proxy rule which, as was mentioned above, induces first-price incentives to bidders with low valuations. In contrast with the proxy rule, a high-type local bidder has no incentive to bid truthfully anymore since her bid affects the price considerably. It is worth highlighting a nice linear functional form of the equilibrium bids in case of nearest-Vickrey rule where correlation parameter γ defines the slope and intercept coefficients. For example, a local bidder shades uniformly across all values when there is no correlation between local bidders, i.e. $\gamma=0$. Positive correlation has an ambiguous effect on revenue and efficiency since it reduces bid-shading for low-type bidders and increases bid-shading for high-type bidders.

Proposition 3. The equilibrium bid function of local bidders under the Proportional Rule is given by:

$$\beta(v)=\begin{cases}0 & v\le d(\gamma)\\ k(\gamma)v-d(\gamma) & v>d(\gamma)\end{cases}\quad \text{if}\quad \gamma<1\qquad \text{and}\qquad \beta(v)=\frac{2}{3}v\quad \text{if}\quad \gamma=1,$$

$$\text{where}\qquad k(\gamma)=\frac{2}{2+\gamma}\qquad d(\gamma)=\frac{2\left(\sqrt{-\gamma^2+2\gamma+8}-3\right)}{\gamma^2+\gamma-2}>0\quad \forall\gamma<1.$$

Proof: See Appendix A.

Surprisingly, the equilibrium bid strategies for our model are the same under the nearest-Vickrey pricing rule and the proportional pricing rule. This result is mainly driven by two of our modeling assumptions: uniform distribution of the global bidder's value with the zero lower bound and the number of local bidders. In a model with more than two items for sale (discussed in greater detail in Section 5), this rule results in different equilibrium bid functions.

Proposition 4. The equilibrium bid function of local bidders under the Nearest-Bid Rule is given by:

$$\beta(v)=\frac{1}{1-\gamma}\left[\ln(2)-\ln(2-(1-\gamma)v)\right]\quad \text{if}\quad \gamma<1\qquad \text{and}\qquad \beta(v)=\frac{1}{2}v\quad \text{if}\quad \gamma=1.$$

---

[21] A symmetric Bayesian-Nash equilibrium for Nearest-Vickrey Rule with local bidders having independent values, i.e. $\gamma = 0$, was derived independently in Goeree and Lien (2011)

Proof: See Appendix A.

Figure 3 (right panel) demonstrates examples of equilibrium bid functions for the nearest-bid rule. The bidding behavior under this rule is very different from the rules already considered. First, the equilibrium bid functions are strictly increasing for all correlation levels. The absence of the zero-bid interval for low-type local bidders is easily explained by the nature of the nearest-bid rule. Conditional on winning, the expected payment of a low-type local bidder is close to zero since the half of the refund to which the bidder is entitled almost surely exceeds the amount of her bid. At the same time, a high-type bidder shades substantially, since her payment depends heavily on the amount of her bid. Second, correlation has a strong negative impact on the equilibrium bidding functions.

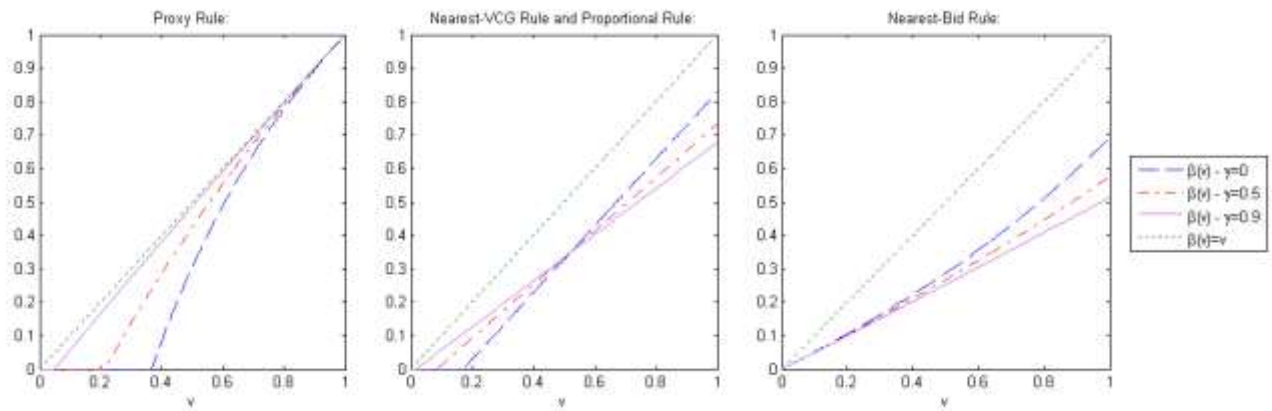

Figure 3. Equilibrium Bids: Proxy Auction Rule (left), Nearest-Vickrey (center) and Nearest-Bid (right)

Figure 2, already seen in the Introduction, summarizes the expected revenue and efficiency results for all pricing rules. The revenue of the VCG rule is negatively affected by positive correlation. By contrast, the performance of the proxy rule improves rapidly as the correlation increases, allowing the proxy rule to outperform the VCG rule in terms of revenue for a substantial range of γ values. Moreover, the proxy rule achieves full efficiency when the locals’ values are perfectly correlated. The performance of the nearest-Vickrey rule seems to be robust to correlation. This suggests that the seller interested in stable revenue and efficiency outcome across different correlation levels might have a good reason to use the nearest-Vickrey rule. However, for this particular model and distributions the proxy rule dominates other core-selecting rules including nearest-Vickrey rule. The performance of the nearest-bid rule falls with correlation, which makes the rule inferior and impractical for this environment. Corresponding numbers for revenue, efficiency and profits of bidders can be found in Table 1. Expressions used to compute revenue and efficiency for all pricing rules are provided in Appendix A. We use simulation results from Baranov (2013) in order to compare all mechanisms considered here with the first-price package auction. Interestingly, the first-price package auction consistently beats the nearest-Vickrey and nearest-bid formats in terms of both revenue and efficiency for any positive correlation value. However, its expected revenue is lower than that of VCG and proxy auction for low and high correlation values respectively.

γ	Statistics	VCG	Proxy	Nearest-VCG	Nearest-Bid	First-Price [22]
$\gamma = 0$	Revenue	0.5833	0.5360	0.5327	0.5	0.5471
	Efficiency	1	0.8679	0.8431	0.8069	0.8754
	Profit Global	0.2916	0.4642	0.4673	0.5	0.4267
	Profit Local	0.2087	0.1342	0.1335	0.1253	0.1498
$\gamma = 0.5$	Revenue	0.5417	0.5852	0.52	0.4521	0.5414
	Efficiency	1	0.9261	0.8356	0.7739	0.9036
	Profit Global	0.3126	0.4148	0.4798	0.5479	0.4297
	Profit Local	0.2295	0.1523	0.1415	0.1252	0.1649
$\gamma = 1$	Revenue	0.5	0.6667	0.5185	0.4167	0.5411
	Efficiency	1	1	0.8334	0.75	0.9049
	Profit Global	0.3335	0.3335	0.4816	0.5834	0.4304
	Profit Local	0.2499	0.1666	0.1481	0.125	0.1757

Table 1. Revenue, Efficiency and Profits

5 Some Extensions

5.1 Non-Uniform Model

The main results of this paper were derived under the assumption of uniform distributions for bidders' values. In this subsection we consider a more general model where the underlying distribution for the local bidders' values allows varying the full-efficiency frequency of winning between the global bidder and local bidders. Specifically, we assume that the cumulative distribution function for local bidders is $F(v) = v^{\alpha}, \alpha > 0$ on the interval $[0,1]$. We continue to assume that the values of the local bidders are perfectly correlated with probability γ and that the global bidder draws her value independently from the uniform distribution on $[0, 2]$.

The parameter α of the local bidders' distribution function can be interpreted in the following way. When α is less than one, the sum of the local bidders' values is expected to be small in comparison with the expected value of the global bidder, implying that the local bidders lose more frequently under full efficiency. When α is greater than one, the situation is reversed, with the global bidder winning less frequently under truthful bidding. In other words, a high α makes the local bidders the stronger side in terms of their expected value.

In general, there are no closed-form solutions for this model, but it can be easily solved by appropriate numerical methods. For example, the equilibrium bidding function for the nearest-Vickrey rule, as in the uniform model, is linear with the slope coefficient being derived explicitly while the intercept term is determined from a non-linear equation which can be solved by a standard numerical procedure like the Newton method. The equilibrium bidding functions for the proxy rule and nearest-bid rule can be easily approximated by numerical methods for solving ODEs. Appendix B contains some equilibrium bidding functions and corresponding equations for numerical approximations for all pricing rules and all correlation levels.

[22] Simulations are based on numerical solutions developed in Baranov (2013)

For the various second-price-like core-selecting auction formats, an increase in α leads to an increase in bid shading by the local bidders. Intuitively, a local bidder expects a higher bid from the other local bidder and tries to free-ride, reducing her bid accordingly. Symmetrically, smaller α results in more truthful bidding since opportunities for free-riding are reduced.

Figures 4 and 5 display revenue and efficiency graphs for scenarios where $\alpha = 2$ and $\alpha = ½$, respectively. As can be seen in Appendix B, the calculations for $\alpha = 2$ for the proxy auction and nearest-bid rule are based on explicit closed-form solutions and for the nearest-VCG rule are based on "almost-closed-form" solutions. Meanwhile, the value $\alpha = ½$ was chosen for symmetric comparison with $\alpha = 2$; most of the associated calculations are based on numerical simulations. Even though a low value for α generates more sincere bids by local bidders, the expected total bid from them is smaller than in case of $\alpha = 1$ (uniform distribution) and so expected seller revenue is lower. The seller revenue is affected positively by an increase in α for all core-selecting rules. On the other hand, an increase in α negatively affects revenue of the VCG auction since it leads to an increase in the probability of low revenue and zero revenue outcomes. As can be seen from Figures 4 and 5, the revenue performance of Vickrey rule relative to any core-selecting rule falls with α. For example, the proxy rule and the nearest-Vickrey rule generate higher revenues for any correlation level $(\forall \gamma)$ when $\alpha = 2$.

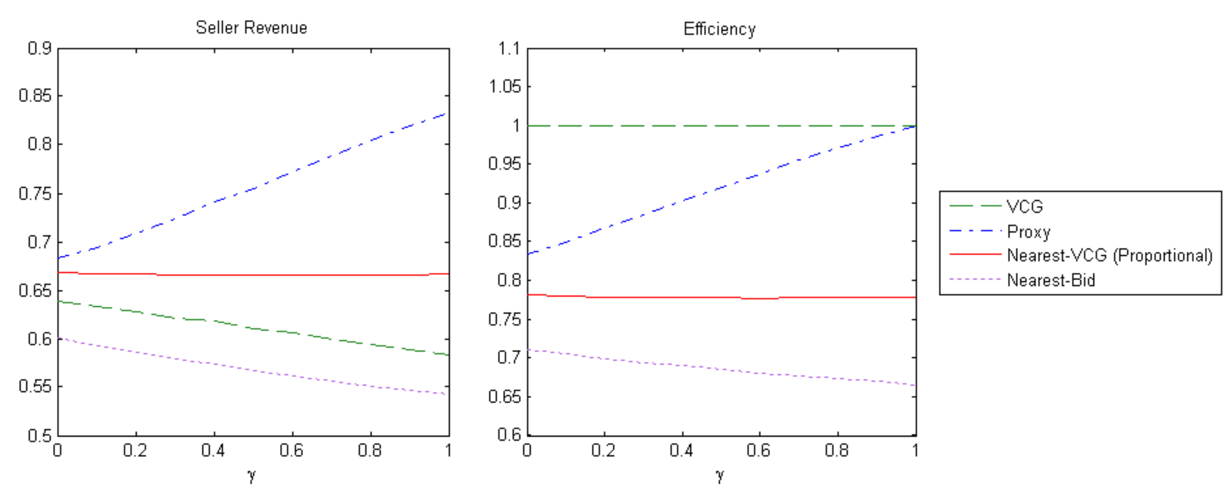

Figure 4. Seller Revenue and Efficiency for $\alpha = 2$ and all $\gamma \in [0,1]$

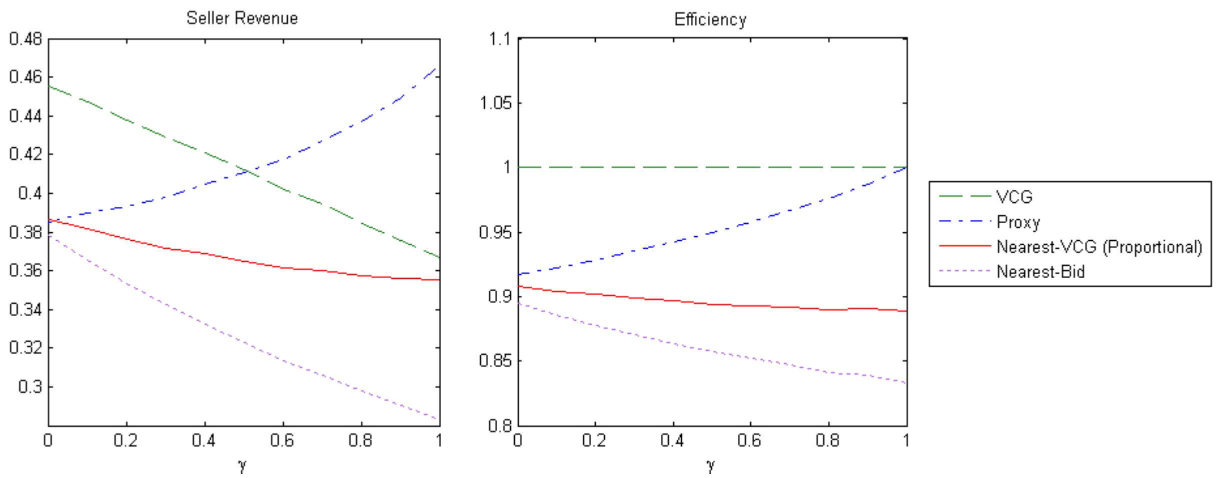

Figure 5. Seller Revenue and Efficiency for α = ½ and all γ ∈ [0, 1]

5.2 Number of Bidders

Here we look into a question of robustness of our results with respect to an increase in the number of bidders. There are several interesting modifications of our model one can consider.

First of all, an increase in the number of global bidders can be modeled as a replacement of the distribution function of the global bidder with the extreme value distribution function of values. For example, if there are two global bidders who draw their values from distribution described by a function $H(u)$ independently from each other and local bidders, a version of the model with one global bidder can be used instead with distribution function of the global bidder being equal to the product of individual distributions, i.e., $G(u) = H^2(u)$. This replacement works because global bidders still have a weakly dominant strategy to bid truthfully. Since the distribution of this pseudo global bidder is no longer uniform, the equilibrium bidding functions of local bidders in general have to be approximated numerically.

Another interesting comparative statics exercise is to increase the number of local bidders together with the number of items offered for sale. For example, consider an auction where three items are offered, with three local bidders who only wish to acquire one item each and a global bidder who is interested only in winning all three items. Keeping a similar value structure, it is possible to solve this model for some correlation levels and some pricing rules. For example, for the proportional rule this model can be solved in closed form for all levels of correlation among local bidders' values. Unfortunately, some pricing rules such as the nearest-Vickrey rule become inherently complex in this environment. Luckily, the solution for the proportional rule sheds some light on the revenue and efficiency performance of the core-selecting rules relative to that of the Vickrey rule. An increase in the number of local bidders leads to a more severe coordination problem among them, more bid shading, and lower seller revenues and efficiency in comparison with the Vickrey rule. This finding suggests that any core-selecting rule may be a poor choice for environments where the presence of a coordination problem is significant.

Finally, one can think of increasing the number of local bidders without increasing the number of products offered in the auction. In such environments, local bidders face competitors for their own item or market and they bid more aggressively. For example, a zero-bid interval (interval of values for which local bidder submits zero bid) will no longer exist in such environment.

5.3 Robustness Check

In this subsection we demonstrate numerically that the partial correlation model for local bidders' values used in this paper results in equilibrium bidding functions which are qualitatively very general. Consider the following modification to the original model of Section 2. Let M be a common unknown distributional factor for local bidders which is distributed on the interval $[0,1]$ with some positive density $f_M(m)$. Conditional on a particular realization, m, of the distributional factor, values for local bidders are drawn independently from a truncated logit distribution on $[0,1]$ with parameters (m,σ), where $\sigma > 0$ is a known scale factor, i.e.:

$$f_L(v_j \mid M = m) = \frac{A(m,\sigma)e^{-(v_j - m)/\sigma}}{\sigma\left(1+e^{-(v_j-m)/\sigma}\right)^2} \quad \text{for } v_j \in [0,1],$$

where $A(m,\sigma)$ is a normalizing constant.

Since m is not observable, both local bidders make inferences about the distribution of the other local bidder's value using their own values as signals about m. It is not hard to show that the conditional density takes the following form:

$$f_L(v_j \mid v_i = s) = \frac{\int_0^1 f_L(v_j \mid m) f_L(s \mid m) f_M(m) dm}{\int_0^1 f_L(s \mid m) f_M(m) dm}$$

Figure 6 contains approximations of conditional densities for different values of the signal and different levels of parameter σ which controls the correlation in this model (taking the role of γ in the main model of this paper). Levels of σ are chosen such that the correlation between local bidders' values in the main model with gamma values 0, 0.5 and 0.9 and the model considered here are approximately equal to each other.

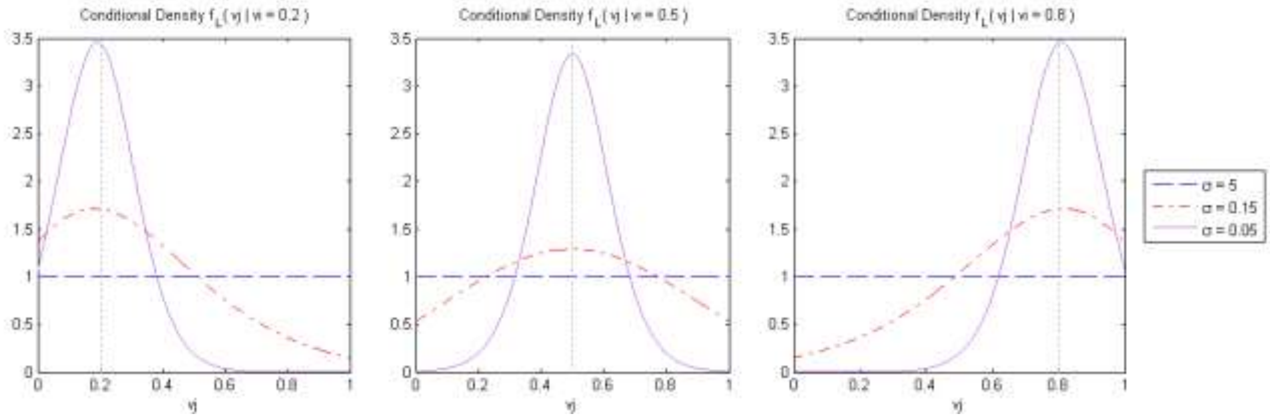

Figure 6. Conditional Densities

Given conditional densities, we approximate first-order conditions for different pricing rules derived in Lemma 3. The corresponding equilibrium bidding functions can be found at Figure 7. The numerical solutions exhibit qualitatively similar shapes and patterns as the closed-form equilibrium bidding functions derived in Section 4 (i.e., Figure 3). These results are very encouraging for the future use of this paper's partial correlation model in other contexts, since in some environments it allows us to generate closed-form solutions or extremely stable and easy numerical solutions without introducing any qualitative distinctions from smoother and more plausible partial-correlation models.

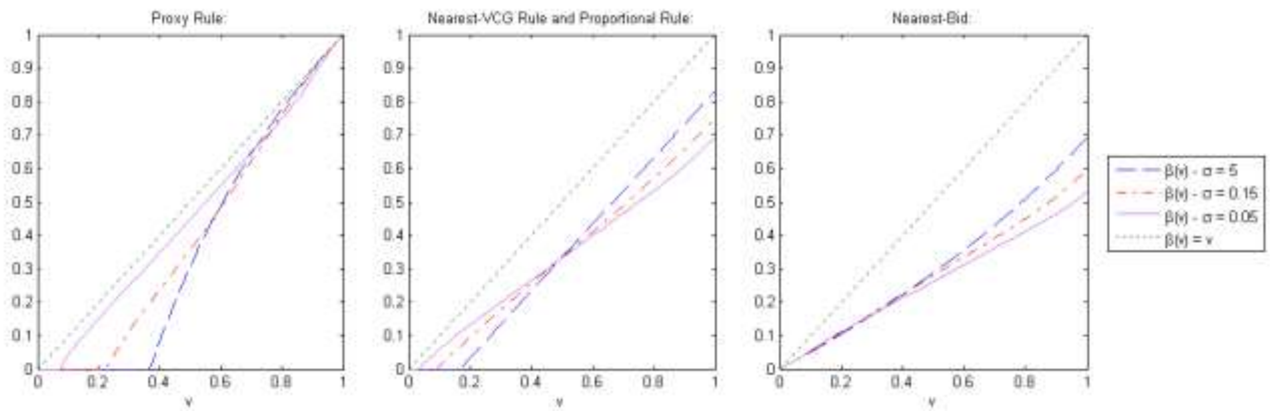

Figure 7. Approximations of Equilibrium Bids

6 Weighted Nearest-VCG Rule

In real-world applications, the issue has arisen recently of how to divide the core burden fairly when the local bidders are making asymmetric purchases. For example, "East" might be twice the value of "West". The rules for spectrum auctions in Australia and Canada have adopted a weighted Nearest-VCG approach, in which the core burden is divided in proportion to the reserve prices of the bidders' packages.[23]

In this section we introduce a simple modification to our model that provides a rationale for the use of exogenous weights. The only change from the main model is to the value generation process for local bidders. The base values, $v_i^b (i=1,2)$, are drawn in the same way as before. However, the actual bidder values are calculated by appropriately scaling the base values: $v_1 = d v_1^b$ and $v_2 = (2-d) v_2^b$, where $d \in (0,2)$. Parameter d controls the degree of asymmetry between local bidders. Note that the overall balance between local bidders and the global bidder is preserved under this modification. We further assume the same uniform distributions that were used in the main part of the paper.

In case of winning, each local bidder pays her VCG payment plus a weighted share of the required core adjustment payment: $p_1 = V_1 + w(B - V_1 - V_2)$ and $p_2 = V_2 + (1-w)(B - V_1 - V_2)$. The asymmetry of local bidders makes closed forms problematic. However, it is still possible to obtain reliable numerical solutions.[24]

Figures 8 and 9 demonstrate the effects of exogenous weights on auction performance. Figure 8 shows that, for symmetric local bidders ($d=1$), symmetric weights are optimal. By contrast, Figure 9 illustrates that, for asymmetric local bidders ($d = 4/3$), asymmetric weights do better—the auction performance can be improved by shifting the core adjustment burden towards the stronger bidder.

Intuitively, when the combination of both local bids are needed to beat the global bid, the VCG price by itself corrects for any bidder asymmetry and therefore equal weights for core-burden allocation are justified.[25] However, when one of the local bids by itself exceeds the global bid, the equal core-burden split puts too much weight on the bidder with smaller VCG payoff, which in turn leads to increased shading.[26] With ex ante bidder value asymmetry, the last argument applies more frequently to a bidder with a weaker value distribution. Exogenously shifting some of the burden towards the stronger bidder helps to reduce the overall bid shading and to achieve higher efficiency and revenues.

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[23] Exogenous weights can be incorporated in other pricing rules in the same way.

[24] When local bidder values are fully independent (i.e. $\gamma = 0$), there is an almost-closed-form solution, $\beta_i(v_i) = \max\{0, v_i - c_i\}$ for $i = 1,2$, where the pair of values c_i need to be determined numerically.

[25] Suppose that $b_1 = 10$, $b_2 = 96$, and $B = 100$. Then $\pi_1^{\text{VCG}} = \pi_2^{\text{VCG}} = 6$ and $\pi_1^{\text{N-VCG}} = \pi_2^{\text{N-VCG}} = 3$ where we assume that local bidders bid truthfully.

[26] Suppose that $b_1 = 10$, $b_2 = 106$, and $B = 100$. Then $\pi_1^{\text{VCG}} = 10, \pi_2^{\text{VCG}} = 16$ and $\pi_1^{\text{N-VCG}} = 5, \pi_2^{\text{N-VCG}} = 11$ where we assume that local bidders bid truthfully. Note that the payoff dropped 50% for the local bidder 1 and only 31% for the local bidder 2 as a result of the core adjustment.

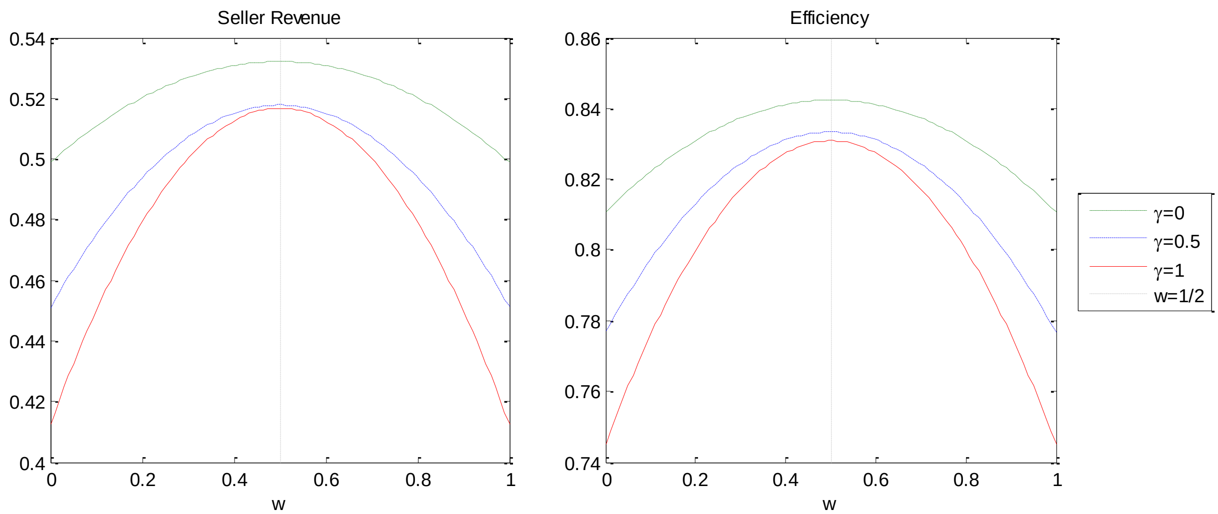

Figure 8. Seller Revenue and Efficiency for $d = 1$ and all $w \in [0,1]$

Our analysis provides partial evidence that the exogenous weights can sometimes be used to mitigate the consequences of the free-rider problem. In general, the weights should help out bidders who are disadvantaged by a particular core-selecting rule. In this spirit, the weighted Nearest-VCG rule used in practice is based on the ratio of the reserve prices (which, in turn, are based on values). However, as seen in Figure 9, it is an open question whether the weights should be proportional to values, or otherwise.

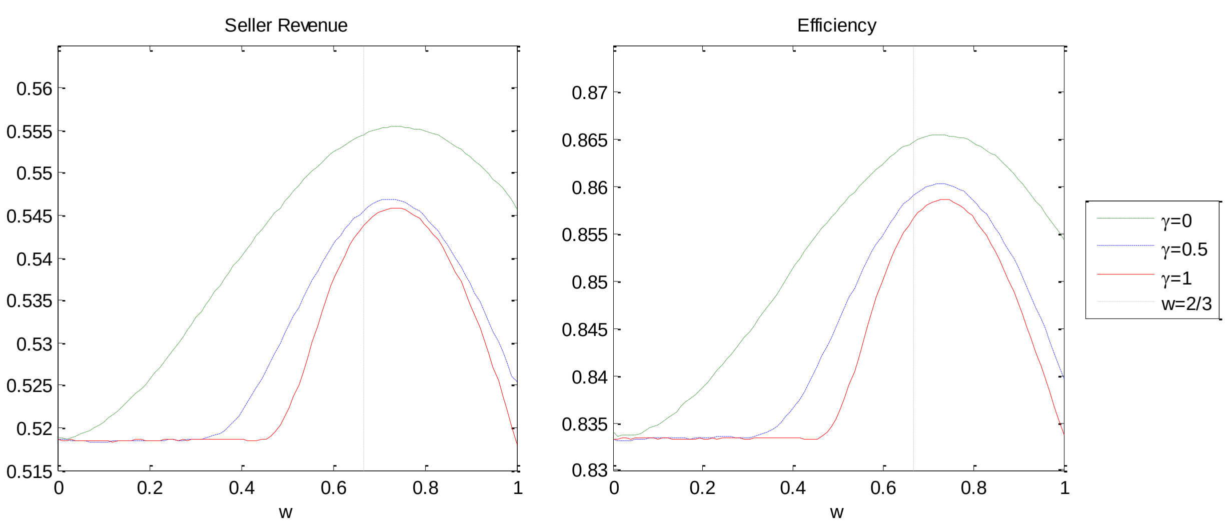

Figure 9. Seller Revenue and Efficiency for $d = 4/3$ and all $w \in [0,1]$

The analysis in this section should be taken with some caution. In fact, one can also construct examples where efficiency is increased when the core burden is completely borne by one of the local bidders, even without asymmetries in value distributions. In such environments, one can produce examples where the weighted Nearest-VCG rule might actually decrease efficiency relative to the "unweighted" version.

7 Conclusion

The past literature has shown the VCG mechanism to have a variety of shortcomings in environments with complementarities, including the possibility of low or even zero revenues, non-monotonicity of revenues with respect to bids and number of bidders, and vulnerability to unusual forms of collusion such as shill-bidding and collusion by losing bidders. This list of drawbacks may help to explain why this auction format — despite its attractive dominant-strategy property — is seldom used in practice: to date, we are not aware of any examples of auctions employing the Vickrey payment rule in an environment with multiple heterogeneous items. At the same time, interest in core-selecting auctions appears to be rising, with ten high-stakes auctions already conducted with a combinatorial clock auction format using the nearest-Vickrey pricing rule. Nevertheless, the existing literature on core-selecting auctions primarily studies complete-information environments and, to the extent that incomplete information is introduced, bidders' values are assumed to be independent.

This paper develops a model of package auctions in an environment with private information. The model considered, while simple and intuitive, includes a number of realistic features that motivate the use of package auctions, such as the presence of substantial complementarities in bidders' preferences and a positive correlation of bidders' values. We were able to derive explicit closed-form solutions for all considered payment rules and all correlation levels, under certain assumptions on distributions.

Our analysis shows that core-selecting payment rules create strong incentives for bidders without a conflict of interests to shade their bids in equilibrium. At first glance, this equilibrium property might discourage the use of core-selecting auctions, since they do not achieve full efficiency and their expected revenue might be even smaller than the revenue of the corresponding VCG auction. However, the presence of positive correlation may dramatically improve the performance of core-selecting auctions relative to the VCG mechanism. In fact, positive correlation significantly improves the performance of the proxy rule while affecting negatively the performance of the VCG. The nature of the proxy rule makes shading profitable only if the local bidder expects a sufficiently high complementary bid from the other local bidder, which becomes increasingly unlikely as the correlation increases. On the other hand, positive correlation increases the probability of low-revenue or zero-revenue outcomes in the VCG mechanism — these occur when *both* local bidders' values are a substantial fraction of the global bidder's value.

Furthermore, the VCG mechanism has other potential drawbacks for practical applications. The standard efficiency properties of the VCG mechanism rely on an assumption that bidders cannot bid under multiple identities ("shill" bidders). If the seller has no control over bidder identities, the efficiency and revenue of the VCG may be significantly impaired. For example, in a simple model with two items, two global bidders and a VCG pricing rule, the truth-telling strategies no longer form equilibria when one or both global bidders can enter the auction using shills. Moreover, in important applications such as spectrum auctions, it is very likely that substantial correlations in bidders' valuations may be present, so the arguments made in the previous paragraph become particularly relevant. Thus, unlike Goeree and Lien (2013), we conclude that there may be good reasons for policymakers to select a core-selecting auction rather than a VCG mechanism.

A curious reader might notice that our paper only considers the case of positive correlation between local bidders' values without considering the case of negative correlation. This treatment seems to be satisfactory since we are not aware of any reasonable practical application for an auction model with negative correlations. The received wisdom in auction theory is that the higher the value of the object to one bidder, the higher the value of the object to any other bidder. Nevertheless, from a methodological viewpoint, a similar model of negative correlation between local bidders' values can be easily constructed. We envision that the presence of negative correlations will improve the performance of the VCG and nearest-bid pricing rules while hurting the proxy rule. This conclusion is based on the intuition developed in Section 4 on the effect of increasing the correlation, only applied in the opposite direction.

References

Ausubel, Lawrence M. (2006), "An Efficient Dynamic Auction for Heterogeneous Commodities," *American Economic Review*, Vol. 96, No. 3, pp. 602-629.

Ausubel, Lawrence M., Peter Cramton, R. Preston McAfee, and John McMillan (1997), "Synergies in Wireless Telephony: Evidence from the Broadband PCS Auctions," *Journal of Economics and Management Strategy*, 6:3, 497-527.

Ausubel, Lawrence M., Peter Cramton and Paul Milgrom (2006), "The Clock-Proxy Auction: A Practical Combinatorial Auction Design," in Peter Cramton, Yoav Shoham, and Richard Steinberg (eds.), *Combinatorial Auctions,* Chapter 5, 115-138, MIT Press.

Ausubel, Lawrence M. and Paul R. Milgrom (2002), "Ascending Auctions with Package Bidding," *Frontiers of Theoretical Economics*, 1(1), Article 1.

Ausubel, Lawrence M. and Paul R. Milgrom (2006), "Ascending Proxy Auctions", Chapter 3 in P. Cramton, Y. Shoham, and R. Steinberg (eds.), *Combinatorial Auctions*, 79-98, MIT Press, 2006.

Beck, Marissa and Marion Ott (2013), "Incentives for Overbidding in Minimum-Revenue Core-Selecting Auctions," *Working Paper*, Stanford and RWHT Aachen University.

Baranov, Oleg (2013), "Exposure vs. Free-Riding in Auctions with Incomplete Information," *Working Paper*, University of Colorado.

Bernheim, B. Douglas and Michael Whinston (1986), "Menu Auctions, Resource Allocation and Economic Influence," *Quarterly Journal of Economics*, 101: 1-31.

Chernomaz, Kirill and Dan Levin (2012), "Efficiency and Synergy in a Multi-Unit Auction with and without Package Bidding: an Experimental Study", *Games and Economic Behavior,* 76:2, pp. 611-635.

Clarke, Edward H (1971). "Multipart Pricing of Public Goods." *Public Choice*, 11, pp. 17–33.

Cramton, Peter (2013), "Spectrum Auction Design", *Review of Industrial Organization*, 42:2, pp. 161-190

Crémer, Jacques and Richard P. McLean (1985), "Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist when Demands are Interdependent," *Econometrica*, 53(2): 345-361.

Day, Robert W. and Cramton, Peter (2012), "The Quadratic Core-Selecting Payment Rule for Combinatorial Auctions," *Operations Research*, 60:3, pp. 588-603.

Day, Robert W. and Milgrom, Paul R (2008), "Core-selecting Package Auctions," *International Journal of Game Theory*, 36, 393-407.

Day, Robert W. and S. Raghavan (2007), "Fair Payments for Efficient Allocations in Public Sector Combinatorial Auctions," *Management Science*, 53, 1389-1406.

Erdil, Aytek and Klemperer, Paul (2010), "A New Payment Rule for Core-Selecting Package Auctions," *Journal of the European Economic Association*, 8:2-3, pp. 537-547.

Fox, Jeremy T. and Patrick Bajari (2013), "Measuring the Efficiency of an FCC Spectrum Auction," *AEJ: Microeconomics, AEA*, vol. 5(1), pp. 100-146.

Goeree, Jacob K. and Yuanchuan Lien (2013), "On the Impossibility of Core-Selecting Auctions," *Theoretical Economics* (accepted subject to revision).

Groves, Theodore (1973), "Incentives in Teams," *Econometrica*, 41(4), pp. 617–31.

Hoffman, Karla, Dinesh Menon, Susara van den Heever and Tomas Wilson (2006), "Observations and Near-Direct Implementations of the Ascending Proxy Auction," in Peter Cramton, Yoav Shoham, and Richard Steinberg (eds.), *Combinatorial Auctions*, Chapter 17, 415-450, MIT Press

Krishna, Vijay and Robert Rosenthal (1996), "Simultaneous Auctions with Synergies," *Games and Economic Behavior*, 17, No. 1, pp. 1-31.

McAfee, Preston R. and Philip J. Reny (1992), "Correlated Information and Mechanism Design," *Econometrica*, 60, No. 2, pp. 395-421.

Milgrom, Paul R. and Robert J. Weber (1982), "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, No. 5, pp. 1089-1122.

Parkes, David C. (2001), *Iterative Combinatorial Auctions: Achieving Economic and Computational Efficiency*, Ph.D. thesis, Department of Computer and Information Science, University of Pennsylvania. http://www.eecs.harvard.edu/~parkes/diss.html.

Parkes, David C. and Ungar, Lyle H. (2000) "Interative Combinational Auctions: Theory and Practice," in *American Association of Artificial Intelligence, ed., Proceedings of the 17th National Conference on Artificial Intelligence (AAAI 2000)*. Menlo Park: AAI Press, pp. 74–81.

Sano, R. (2011), "Incentives in Core-Selecting Auctions with Single-Minded Bidders," *Games and Economic Behavior*, 72(2), pp. 602-606.

Vickrey, William (1961), "Counterspeculation, Auctions, and Competitive Sealed Tenders." *Journal of Finance*, 16(1), pp. 8–37.

Appendix A – Proofs

Proof of Lemma 4:

The profit function of a local bidder i is given by:

$$\pi_i(b_i, v_i) = v_i \Phi_i(b_i, v_i) - P_i(b_i, v_i)$$

The first-order optimality conditions are:

$$\frac{\partial \pi_i(b_i^*, v_i)}{\partial b_i} \le 0 \quad b_i^* \ge 0 \quad b_i^* \left(\frac{\partial \pi_i(b_i^*, v_i)}{\partial b_i} \right) = 0 \quad \forall v_i \in [0,1]$$

In order to get the desired form of the first-order conditions, we first compute the marginal cost of winning with a bid b_i. Using the continuous pricing property, we can differentiate the expected payment with respect to b_i:

$$\begin{aligned}
\frac{\partial P_i(b_i, v_i)}{\partial b_i} &= \frac{\partial}{\partial b_i} \left(\gamma \int_{b_i + \beta(v_i) > u} p_i(b_i, \beta(v_i), u) g(u)\, du + (1-\gamma) \iint_{b_i + \beta(v_j) > u} p_i(b_i, \beta(v_j), u) f(v_j) g(u)\, dv_j du \right) \\
&= \gamma p_i(b_i, \beta(v_i), b_i + \beta(v_i)) g(b_i + \beta(v_i)) + (1-\gamma) \int_{v_j} p_i(b_i, \beta(v_j), b_i + \beta(v_j)) f(v_j) g(b_i + \beta(v_j)) dv_j \\
&\quad + \gamma \int_{b_i + \beta(v_i) > u} p_i'(b_i, \beta(v_i), u) g(u)\, du + (1-\gamma) \iint_{b_i + \beta(v_j) > u} p_i'(b_i, \beta(v_j), u) f(v_j) g(u)\, dv_j du \ .
\end{aligned}$$

Note that by Lemma 1 and the pivotal pricing property we have the following:

$$p_i(b_i, \beta(v_i), b_i + \beta(v_i)) = p_i(b_i, \beta(v_j), b_i + \beta(v_j)) = b_i \quad \forall v_i, v_j$$

Plugging the last equality back to the marginal cost of winning we get a short and intuitive form for this term:

$$\begin{aligned}
\frac{\partial P_i(b_i, v_i)}{\partial b_i} &= b_i \left[\gamma g(b_i + \beta(v_i)) + (1-\gamma) \int_{v_j} f(v_j) g(b_i + \beta(v_j)) dv_j \right] + MP_i(b_i, v_i) \\
&= b_i \varphi_i(b_i, v_i) + MP_i(b_i, v_i)
\end{aligned}$$

Finally, the desired form of the first-order optimality conditions:

$$\frac{\partial \pi_i(b_i^*, v_i)}{\partial b_i} = v_i \varphi_i(b_i^*, v_i) - \frac{\partial P_i(b_i, v_i)}{\partial b_i} = \left(v_i - b_i^*\right) \varphi_i(b_i^*, v_i) - MP_i(b_i^*, v_i)$$

$$\left(v_i - b_i^*\right) \varphi_i(b_i^*, v_i) - MP_i(b_i^*, v_i) \le 0 \quad b_i^* \ge 0$$

$$b_i^* \left[\left(v_i - b_i^*\right) \varphi_i(b_i^*, v_i) - MP_i(b_i^*, v_i) \right] = 0 \quad \forall v_i \in [0,1]$$

Proof of Proposition 1:

The following table summarizes marginal payments for a local bidder in all possible situations:

$\beta(v_i), \beta(v_j)$	$u \le 2\underline{\beta}$	$2\underline{\beta} < u \le \beta(v_i) + \beta(v_j)$	$u > \beta(v_i) + \beta(v_j)$
Perfect Correlation $v_i = v_j$	0	N/A	0
Independence $v_i < v_j$	0	1	0
Independence $v_i > v_j$	0	0	0

(* $\underline{\beta} = \beta(\min(v_i, v_j))$)

The expected marginal payment for a local bidder in equilibrium is:

$$MP(\beta(v), v) = \frac{(1-\gamma)}{2} \int_v^1 \left(\beta(v_j) - \beta(v)\right) dv_j .$$

$$(v - \beta(v)) = (1-\gamma) \int_v^1 \left(\beta(v_j) - \beta(v)\right) dv_j .$$

Note that $\beta(v) = v$ in case of $\gamma = 1$. The associated expected revenue and efficiency are

$$R^{\text{Proxy}} = \tfrac{2}{3}, \quad Ef^{\text{Proxy}} = 1 .$$

For the case of $\gamma < 1$, the equivalent differential equation and the terminal condition are given by:

$$\beta' = \frac{1}{\gamma + (1-\gamma)v} \quad \beta(1) = 1 .$$

The solution for this differential equation yields the equilibrium bid function.
The associated expected revenue and efficiency are given by:

$$\left.\begin{aligned} R^{\text{Proxy}} &= \frac{6e^{-(1-\gamma)} - e^{-2(1-\gamma)} - (1 + 5\gamma - 2\gamma^2 + \gamma^3)}{2(1-\gamma)^3} \\ Ef^{\text{Proxy}} &= \frac{2e^{-(1-\gamma)} + (1 - 4\gamma + \gamma^2)}{2(1-\gamma)^2} \end{aligned}\right\} \quad \forall \gamma < 1$$

Proof of Proposition 2:

The following table summarizes marginal payments for a local bidder in all possible situations:

$\beta(v_i), \beta(v_j)$	$u \le \beta(v_i)$	$\beta(v_i) < u \le \beta(v_i) + \beta(v_j)$	$u > \beta(v_i) + \beta(v_j)$
Perfect Correlation $v_i = v_j$	0	½	0
Independence	0	½	0

The expected marginal payment for a local bidder in equilibrium is:

$$MP(\beta(v),v)=\frac{1}{4}\left[\gamma\beta(v)+(1-\gamma)\int_0^1\beta(v_j)dv_j\right]$$

$$(v-\beta(v))=\frac{1}{2}\left[\gamma\beta(v)+(1-\gamma)\int_0^1\beta(v_j)dv_j\right]$$

Note that $\beta(v)=2/3\,v$ in case of $\gamma=1$. The associated expected revenue and efficiency are:

$$R^{\text{Nearest-VCG}}=14/27 \quad Ef^{\text{ Nearest-VCG}}=5/6$$

If $\gamma<1$:

$$\beta(v)=\frac{2}{2+\gamma}\left[v-\frac{(1-\gamma)}{2}\int_0^1\beta(v_j)dv_j\right]=k(\gamma)v-d(\gamma)$$

The associated expected revenue and efficiency are given by:

$$R^{\text{Nearest-VCG}}=\frac{(d-1)\left(-3d^3(1-\gamma)(1+\gamma)^2+13d^2(1+\gamma)+d(7+25\gamma+12\gamma^2)-11\gamma-17\right)}{6(2+\gamma)^2},$$

$$Ef^{\text{ Nearest-VCG}}=\frac{2d^2(1+\gamma)-2d(2+\gamma)+\gamma+4}{2(2+\gamma)}$$

where $d(\gamma)=\dfrac{2\left(\sqrt{-\gamma^2+2\gamma+8}-3\right)}{\gamma^2+\gamma-2}$ and $\gamma<1$.

Proof of Proposition 3:

The following table summarizes marginal payments for a local bidder in all possible situations:

$\beta(v_i),\beta(v_j)$	$0<u\le\beta(v_i)+\beta(v_j)$	$u>\beta(v_i)+\beta(v_j)$
Perfect Correlation $v_i=v_j$	$\frac{\beta(v_i)u}{4\beta(v_i)^2}$	0
Independence	$\frac{\beta(v_j)u}{\left(\beta(v_i)+\beta(v_j)\right)^2}$	0

The expected marginal payment for a local bidder in equilibrium is:

$$MP(\beta(v),v)=\frac{1}{4}\left[\gamma\beta(v)\frac{\left(2\beta(v)\right)^2}{4\beta(v)^2}+(1-\gamma)\int_0^1\beta(v_j)\frac{\left(\beta(v)+\beta(v_j)\right)^2}{\left(\beta(v)+\beta(v_j)\right)^2}dv_j\right]=\frac{1}{4}\left[\gamma\beta(v)+(1-\gamma)\int_0^1\beta(v_j)dv_j\right]$$

Note that the expected marginal payment is exactly the same as the one for Nearest-VCG Payment Rule. Therefore, equilibrium bid function, expected revenue and efficiency are exactly the same.

Proof of Proposition 4:

The following table summarizes marginal payments for a local bidder in all possible situations:

$\beta(v_i),\beta(v_j)$	$u\le\overline{\beta}-\underline{\beta}$	$\overline{\beta}-\underline{\beta}<u\le\beta(v_i)+\beta(v_j)$	$u>\beta(v_i)+\beta(v_j)$
Perfect Correlation $v_i=v_j$	N/A	½	0
Independence	0	½	0

(* $\underline{\beta}=\beta(\min(v_i,v_j))$ $\overline{\beta}=\beta(\max(v_i,v_j))$)

The expected marginal payment for a local bidder in equilibrium is:

$$MP(\beta(v),v)=\frac{\gamma}{2}\beta(v)+\frac{(1-\gamma)}{2}\left[\int_0^v\beta(v_j)dv_j+\int_v^1\beta(v)dv_j\right]=\frac{\gamma}{2}\beta(v)+\frac{(1-\gamma)}{2}\left[\int_0^v\beta(v_j)dv_j+\beta(v)(1-v)\right]$$

$$(v-\beta(v))=\gamma\beta(v)+(1-\gamma)\left[\int_0^v\beta(v_j)dv_j+\beta(v)(1-v)\right]$$

Note that $\beta(v)=\frac{1}{2}v$ in case of $\gamma=1$. The associated expected revenue and efficiency are:

$$R^{\text{Nearest-Bid}}=5/12 \quad Ef^{\text{Nearest-Bid}}=3/4$$

For the case $\gamma<1$ the equivalent differential equation and initial condition are given by:

$$\beta'=\frac{1}{2-(1-\gamma)v} \quad \beta(0)=0$$

The associated expected revenue and efficiency are given by:

$$R^{\text{Nearest-Bid}}=\frac{1+\gamma\left(\gamma(5+4\ln 2)-6+4\ln 2\right)-4\gamma(1+\gamma)\ln(1+\gamma)}{2(1-\gamma)^3}$$

$$Ef^{\text{Nearest-Bid}}=\frac{3-4\gamma+\gamma^2+2(1+\gamma)\left(\ln\left(1+\gamma\right)-\ln 2\right)}{2(1-\gamma)^2} \qquad \forall\gamma<1$$

Appendix B – Solution Summary

1) Proxy Rule

Local Distribution Parameter $\alpha > 0 \left(F(v) = v^{\alpha}, \alpha > 0\right)$	Correlation Parameter $0 \le \gamma \le 1$	Symmetric Equilibrium Bid
$\alpha > 0$	$\gamma = 1$	$\beta(v) = v$
$\alpha = 1$	$0 \le \gamma < 1$	$\beta(v) = \max\left(0, 1 + \frac{\ln(\gamma + (1-\gamma)v)}{1-\gamma}\right)$
$\alpha \ne 1$	$\gamma = 0$	$\beta(v) = \max\left(0, \frac{v^{1-\alpha} - \alpha}{1-\alpha}\right)$
$\alpha = 2$	$0 < \gamma < 1$	$\beta(v) = \max\left(0, \frac{1}{\sqrt{\gamma(1-\gamma)}} \tan^{-1}\left(\sqrt{\frac{1-\gamma}{\gamma}} v\right) + C\right)$ $C = 1 - \frac{1}{\sqrt{\gamma(1-\gamma)}} \tan^{-1}\left(\sqrt{\frac{1-\gamma}{\gamma}}\right)$
$\alpha \ne 1, \alpha \ne 2$	$0 < \gamma < 1$	No closed-form solution. Differential Equation: $\beta' = \frac{1}{\gamma + (1-\gamma)v^{\alpha}} \quad \beta(1) = 1$

2) Nearest-VCG Rule (Proportional Rule)

Local Distribution Parameter $\alpha > 0 \left(F(v) = v^{\alpha}, \alpha > 0\right)$	Correlation Parameter $0 \le \gamma \le 1$	Symmetric Equilibrium Bid
$\alpha > 0$	$\gamma = 1$	$\beta(v) = \frac{2}{3} v$
$\alpha = 1$	$0 \le \gamma < 1$	$\beta(v) = \begin{cases} kv - d & v > \frac{d}{k} \\ 0 & v \le \frac{d}{k} \end{cases}$ $k = \frac{2}{2+\gamma} \quad d = \frac{3k^2 - 2k\sqrt{3k-1}}{3k-2}$
$\alpha \ne 1$	$\gamma = 0$	Almost-Closed Form: $\beta(v) = \begin{cases} kv - d & v > \frac{d}{k} \\ 0 & v \le \frac{d}{k} \end{cases}$

$\alpha \neq 1$	$0 < \gamma < 1$	$k = \dfrac{2}{2+\gamma}$ d is defined by the equation: $\dfrac{d^{\alpha+1}}{(1+\alpha)k^{\alpha}} - \dfrac{3k}{3k-2}d + \dfrac{\alpha k}{\alpha+1} = 0$ Note that the equation has exactly one root on (0, k) interval

3) Nearest-Bid Rule

Local Distribution Parameter $\alpha > 0$ $\left(F(v) = v^{\alpha}, \alpha > 0\right)$	Correlation Parameter $0 \le \gamma \le 1$	Symmetric Equilibrium Bid
$\alpha > 0$	$\gamma = 1$	$\beta(v) = \frac{1}{2}v$
$\alpha = 1$	$0 \le \gamma < 1$	$\beta(v) = \frac{1}{1-\gamma}\left[\ln(2) - \ln(2-(1-\gamma)v)\right]$
$\alpha = 2$	$0 \le \gamma < 1$	$\beta(v) = \frac{1}{\sqrt{2(1-\gamma)}}\ln\left\lvert\frac{(1-\gamma)v+\sqrt{2(1-\gamma)}}{(1-\gamma)v-\sqrt{2(1-\gamma)}}\right\rvert$
$\alpha \neq 1, \alpha \neq 2$	$\gamma = 0$	No closed-form solution. Differential Equation: $\beta' = \dfrac{1}{2-(1-\gamma)v^{\alpha}} \quad \beta(0) = 0$
$\alpha \neq 1, \alpha \neq 2$	$0 < \gamma < 1$	