Iterative Vickrey Pricing in Dynamic Auctions

Lawrence M. Ausubel* and Oleg Baranov†

Preliminary and Incomplete

August 16, 2018

Abstract

Auction literature offers two prescriptions for achieving efficient outcomes in practical auction settings. First, an auction design should use opportunity-cost pricing to the extent possible to promote truthful revelation of bidder preferences. Second, the pricing mechanism should be implemented via an iterative first-price process where, at each iteration, all bidders are informed of their provisional winning allocations and associated payments. For the general heterogeneous setting with private values, we develop an auction design that adheres to both principles. We demonstrate how novel auction design resolves many of the issues associated with the SMRA and CCA designs—two leading auction formats currently used for spectrum auctions.

*Department of Economics, University of Maryland, College Park, MD 20742, USA. Email: ausubel@econ.umd.edu

†Department of Economics, University of Colorado, Boulder, CO 80309, USA. Email: oleg.baranov@colorado.edu

Over the past two decades, multi-item auction design and practical spectrum auctions have developed hand-in-hand. Internationally, auctions have become the accepted practice for allocating spectrum licenses and, conversely, spectrum auctions have been a key application driving the market design literature. Two auction formats from the market design field have been utilized repeatedly in recent years: the simultaneous multiple-round auction (SMRA); and the combinatorial clock auction (CCA). The older SMRA has been used for essentially all US spectrum auctions from 1994 - 2015, and it or its close variants were used for most of the European 3G auctions (2000 - 2001), spectrum auctions in India (2010 and 2015), and many other auctions worldwide. Meanwhile, the newer CCA has been the most common format in the recent round of digital dividend auctions in Europe, Canada and Australia.

Despite (or, perhaps because of) their ubiquity, both of these auction formats have been subject to numerous critiques. A key weakness of the SMRA is that bidding is on single items, as opposed to packages, giving rise to an "exposure" problem: a bidder with superadditive valuations would be hesitant to bid more than its standalone values for individual spectrum licenses, since it could get stuck winning some licenses but not their complements. The SMRA also suffers from a "demand reduction" problem—uniform pricing induces large bidders to make room for their rivals—and a related "holdup" problem, as a bidder may successively bid up the prices of an opponent's items until the opponent cedes items to the opportunistic bidder. The SMRA's activity rule is artificially based on points, producing strategic opportunities such as "parking". Largely in response to these issues, researchers proposed the CCA, with attempted ways to mitigate the exposure, demand reduction and activity rule issues. Unfortunately, while it fully eliminates the exposure problem, the CCA as currently implemented provides only imperfect solutions to the latter problems, producing new potential opportunities for strategic manipulation.

Most critiques of the CCA may be traced to that it can be classified as an *iterative second-price auction*. In multiple bidding rounds, a bidder submits a series of package bids, (S, p), where S denotes a set of items and p denotes an associated all-or-nothing payment for the set. However, if the bidder ultimately wins S, the payment may be an amount significantly less than the final p for that set, since prices are Vickrey-based. By way of contrast, most other popular dynamic auctions (including the SMRA) are *iterative first-price auction* formats. As the clearest example, consider the English auction for a single item, as practiced by Sotheby's or eBay. While it is often modeled as a sealed-bid second-price auction (William Vickrey, 1961), bidders iteratively submit bids which, if they turn out to win, specify the actual amounts that will be paid. While effectively a second-price auction, it is literally an iterative first-price auction.

One consequence of iterative second-pricing is a mispricing problem. When a bidder wins package S, his payment is based on the opportunity cost of S, which in turn is calculated from (non-winning) bids in which opponents demand the constituent elements of S. There is substantial likelihood that opponents may not have bid for the requisite packages for determining the bidder's payment for S. The CCA partly addresses this issue by allowing a supplementary bidding round in which values for packages may be expressed, but there may be little incentive for bidders to reveal truthfully their price-determining values. On the one hand, if the bid will only benefit the seller and not the bidder, the opponent may not bother to submit the bid, often referred to as a "missing bid" or "quiet bidding" problem. On the other hand, a spiteful bidder may wish gratuitously to increase the opponents' costs and may therefore bid more than its value when the bidder is confident that the bid will be non-winning, often referred to as a "predatory bidding" problem (Jonathan Levin and Andrzej Skrzypacz, 2016). The second-pricing may also be problematic for a budget-constrained bidder. In an iterative first-price auction, a bidder who is not permitted to spend more than B would merely refrain from ever bidding more than B. However, it is unclear how the budget-constrained bidder should bid if the current required bid is greater than B but the anticipated payment is less than B.

A common theme of critiques of both the SMRA and the CCA is that deficiencies in activity rules may be exploited by bidders. The objective of activity rules is to enforce consistency between bidding in early rounds and later rounds, so as to improve the information aggregation and informativeness of the auction, to avoid surprises such as bid sniping, and to mitigate strategic manipulations. The prevalent approach, point-based activity rules, exogenously assigns a number of points to every item. A bidder's "activity", defined as the number of points associated with the bidder's demand, is essentially required to be monotonic from round to round. Critiques have emphasized that point-based rules are both too weak and too strong. They insufficiently constrain by permitting bids on items that the bidder is not interested in buying, for the purpose of stockpiling points for later ("parking"). Simultaneously, they overly constrain in that, for any choice of point values, there exist histories under which a point-based rule prevents straightforward bidding according to true values ("one-way cat flaps"). Most CCAs have avoided over-constraining bidders by introducing a revealed-preference exception; however, in the course of relaxing point-based issues, these hybrid activity rules only exacerbate the under-constraining of bidders. Finally, while it is recognized that the choice of points may affect performance, there is no general guidance as to how they should be chosen.

Our objective in the current paper is to produce a synthesis auction design that combines the most desirable properties of the SMRA and the CCA. From the SMRA, we preserve the property that the bidder knows the payment for its current demanded package, should the auction end right now, i.e., it is effectively an iterative first-price auction. From the CCA, we inherit opportunity-cost-based pricing, so that the auction is not vulnerable to the most overt forms of demand reduction or holdup.

To enable this, we implement an activity rule based upon the Generalized Axiom of Revealed Preference (GARP). This activity rule confirms that each successive bid, interpreted as a quantity choice at the prevailing prices, together with all prior bids, can be rationalized by a utility function (Afriat's Theorem). As such, it is the strictest activity rule that is consistent with truthful bidding by a utility-maximizing individual. In a companion paper (Ausubel and Baranov, 2016), we demonstrate that this activity rule displays a number of properties helpful for practical implementation; for example, it never leads bidders into "dead-ends".

Our method for generating the "payment" for bidder i's currently demanded package (S_i) is as follows. We ask the question, "What is the highest payment that bidder i may be required to pay for S_i, given the bidding history and that its opponents are subject to GARP?" Observe that, at prevailing prices, bidder i's opponents may need to violate GARP in order to purchase S_i in addition to their own demands; however, with a suitable discount, such purchase can be made consistent. We define the GARP violation to be the minimal discount to the current price of S_i that would rationalize its purchase by opponents — and hence the minimal discount that bidder i would receive under opportunity-cost pricing. In the simple environment of a homogeneous good with diminishing marginal values, the GARP violation coincides with the buyer's discount in the multi-unit Vickrey auction (Vickrey, 1961) and with the "clinching" discount (Ausubel, 2004). The contribution of the current methodology is to generalize the older approaches to more general environments with heterogeneous goods that may be substitutes or complements, while providing a practical way that these can be determined and presented to bidders in iterative fashion as the auction progresses.

Our paper relates most directly to two lines of literature. The first concerns dynamic implementations of the Vickrey-Clarke-Groves (VCG) mechanism. Vickrey (1961), Clarke (1971) and Groves (1973) proposed the mechanism with opportunity-cost-based pricing for which truth-telling is a dominant strategy. A long line of subsequent authors have explored obtaining the VCG mechanism or related outcomes through dynamic auction procedures, including Demange et al. (1986), Gul and Stacchetti (2000), Parkes and Ungar (2000, 2002), Ausubel and Milgrom (2002), Bikhchandani and Ostroy (2002, 2006), Ausubel (2004, 2006), de Vries et al. (2007), Mishra and Parkes (2007), Lamy (2012), and Baranov et al. (2017). A survey of this literature can be found in Parkes (2006).

The second related line of literature concerns the design of spectrum auctions. The SMRA was explored by McMillan (1994), Cramton (1995), McAfee and McMillan (1996), Milgrom (2000, 2004), Binmore and Klemperer

(2002), and Bulow, Levin and Milgrom (2017). The CCA was explored by Ausubel, Cramton and Milgrom (2006), Cramton (2013), Bichler et al. (2013, 2014), Ausubel and Baranov (2014), Levin and Skrzypacz (2016), Janssen and Karamychev (2016), Janssen and Kasberger (2017), Gretschko, Knapek and Wambach (2017), and Bichler and Goeree (2017). **The exposure, demand reduction and other issues were studied in ... .**

This paper has the following organization. Section 1 presents an intuitive example of the approach taken to iterative Vickrey pricing. Section 2 specifies a model of the general environment with private values and develops the relevant theory for GARP. A full specification of the proposed auction design and the main results are given in Section 3. Results on bidder incentives can be found in Section 4. Practical implementation issues, including iterative first-pricing, are discussed in Section 5. Section 6 concludes. Appendix A provides an alternative treatment based on the use of aggregate demand information, and all proofs are relegated to Appendix B.

1 Example of Iterative Vickrey Pricing

In this section, we illustrate our iterative pricing approach. Consider an example with two unique goods, A and B, and two bidders. The auctioneer initializes price clocks for both goods at zero. At any time t, the auctioneer quotes current prices $p(t)$, and both bidders reply with their demands. The auctioneer increments clock prices for goods with excess demand. Table 1 provides information about bidders' values, the Vickrey outcome and the detailed bidding history.

At each time t, we ask the following question: "What is the highest payment that Bidder 1 might be required to pay for package AB given that the bidding of Bidder 2 is constrained by GARP?" If Bidder 2 can bid for AB without violating GARP, then Bidder 1's opportunity cost can be as high as the current price. By contrast, if Bidder 2 must violate GARP in order to bid for AB, then Bidder 1 is entitled to a discount since its current opportunity costs are strictly less than the current price.

Table 1: *Example of Dynamic Vickrey Pricing*

	Bidder 1		**Bidder 2**	
Values ($\emptyset$, A, B, AB):	$v_1 = (0, 20, 10, 60)$		$v_2 = (0, 30, 20, 33)$	
Vickrey Outcome:	$x_1^* = AB,\ y_1^V = 33$		$x_1^* = \emptyset,\ y_2^V = 0$	
Clock Prices $p(t)$	Demand	Exposure	Demand	Exposure
$(0, 0) \to (3, 3)$	AB	$0 \to 6$	AB	$0 \to 6$
$(3, 3) \to (13, 3)$	AB	$6 \to 16$	A	$3 \to 13$
$(13, 3) \to (30, 20)$	AB	$16 \to 33$	A or B*	$13 \to 30$*
$p(T) = (30, 20)$	AB	33	$\emptyset$	0

* – Bidder 2 is indifferent between A and B. Exposure for A is reported

First, observe that Bidder 2 has reduced its demand from AB to A at $(3, 3)$, revealing that $v_2(A) - 3 \geq v_2(AB) - 6$. Now consider a moment when prices reach $(13, 3)$. We need to check whether Bidder 2 can bid for AB without violating GARP. To perform this check, we solve for the minimum discount λ that must be applied to package AB to satisfy the following system of revealed preference constraints (as if Bidder 2 were to demand AB at $(13, 3)$):

$$\begin{array}{rcl} v_2(A) - 3 & \geq & v_2(AB) - 6 \\ v_2(A) - 13 & \leq & v_2(AB) - (16 - \lambda) \end{array} \quad \Rightarrow \quad v_2(A) + 3 - \lambda \leq v_2(AB) \leq v_2(A) + 3.$$

The minimum discount λ is zero, and bidding for AB would be consistent with GARP. Thus, Bidder 1's opportunity costs for AB can be as high as 16 (the current price of package AB).

Next, consider a moment when prices reach $(30, 20)$. Similar to the previous calculation, we solve for the minimum discount that satisfies the following system of revealed preference constraints:

$$\begin{array}{rcl} v_2(A) - 3 & \geq & v_2(AB) - 6 \\ v_2(A) - 30 & \leq & v_2(AB) - (50 - \lambda) \end{array} \quad \Rightarrow \quad v_2(A) + 20 - \lambda \leq v_2(AB) \leq v_2(A) + 3.$$

Now the minimum required discount λ is 17, and Bidder 2 would violate GARP by biding for AB at this time. Thus, Bidder 1's opportunity cost for package

AB cannot exceed 33 (calculated as $50 - 17$). Note that 33 is the actual Vickrey payment of Bidder 1.

2 Model

A seller offers multiple units of K heterogeneous indivisible goods, denoted by $S = \{s^1, ..., s^K\} \in \mathbb{Z}^K_{++}$ to a set of bidders $N = \{1, 2, ..., n\}$. The set of all possible bundles of items in S is denoted by $\Omega = \{(z^1, ..., z^K) : 0 \leq z^k \leq s^k \quad \forall k \in \{1, ..., K\}\}$. For every bidder $i \in N$, and every bundle $z \in \Omega$, the valuation of bidder i is given by $v_i(z) \geq 0$. A bidder's value for the null bundle is zero, $v_i(\emptyset) = 0$. We make the following standard assumptions about valuation functions:

(A1) *Pure Private Values:* Each bidder i knows its own valuation for any bundle z, and this valuation does not depend on valuations of other bidders;

(A2) *Quasilinear Values:* The payoff of bidder i from winning bundle z in exchange for a payment y is given by $v_i(z) - y$;

(A3) *Monotonicity:* The value function $v_i(\cdot)$ is weakly increasing in z, i.e., $v_i(z') \geq v_i(z)$ for any $z' \geq z$.

An allocation $x = (x_1, ..., x_n)$ is *feasible* if $x_i \in \Omega$ for all $i \in N$ and $\sum_N x_j \leq S$. Denote X a set of all feasible allocations. The *coalitional value function* for bidders in coalition $M \subseteq N$ is given by:

$$w(M) = \max_{x \in X} \sum_{j \in M} v_j(x_j) \tag{2.1}$$

A feasible allocation $x = (x_1, ..., x_n) \in X$ is *efficient* if $\sum_N v_j(x_j) = w(N)$. Let $N_{-i} = N \backslash \{i\}$ denote the coalition of all bidders in N excluding bidder i. A *Vickrey outcome* consists of an efficient allocation vector $x^* = (x^*_1, ..., x^*_n)$ and a corresponding payment vector $y^V = (y^V_1, ..., y^V_n)$ where $y^V_i = w(N_{-i}) -$

$\sum_{N_{-i}} v_j(x_j^*)$ for all $i \in N$. A Vickrey payoff for bidder i is given by $\pi_i^V = v_i(x_i^*) - y_i^V = w(N) - w(N_{-i})$.

For a general private value setting, a fully dynamic implementation of the Vickrey outcome, such as the one developed by Mishra and Parkes (2007), can be impractical for applications such as spectrum auctions. To avoid the complexity of exact implementation, we approximate the Vickrey outcome using a sequence of two standard auction procedures: a clock auction *(the primary stage)* followed by a sealed-bid auction *(the secondary stage)*. In the primary stage, our design approximates a quasi Vickrey outcome (a generalization of the Vickrey outcome that can be applied to any feasible allocation, not just the efficient one). In the the secondary stage, the approximation from the primary stage is converted into an approximation of the true Vickrey outcome.

Formally, a *quasi Vickrey outcome* consists of a feasible allocation $x = (x_1, ..., x_n) \in X$ and a corresponding payment vector $y^Q(x) = (y_1^Q(x), ..., y_n^Q(x))$ where $y_i^Q(x) = w(N_{-i}) - \sum_{j \in N_{-i}} v_j(x_j)$ for all $i \in N$. A quasi Vickrey payoff for bidder i is then given by $\pi_i^Q(x) = v_i(x_i) - y_i^Q(x)$. Intuitively, a quasi Vickrey outcome is derived using the same familiar calculations as the ones used for the Vickrey outcome, but applied to a feasible allocation x instead of an efficient allocation x^*. Table 2 provides an example of the Vickrey outcome and one of the quasi Vickrey outcomes for a setting with two unique goods, A and B, and three bidders.

Given an efficient allocation x^*, denote $\Delta v_i(z) = v_i(x_i^*) - v_i(z)$ the net value gain of bidder i from bundle x_i^* relative to bundle z. Denote $\Delta v_M(x) = \sum_M \Delta v_j(x_j)$ the joint net value gain of bidders in coalition M from allocation x^* relative to allocation $x = (x_1, ..., x_n)$. Note that $\Delta v_N(x) \geq 0$ by construction since it equals to the total inefficiency of allocation x.

A quasi Vickrey outcome coincides with the Vickrey outcome when the its underlying feasible allocation $x \in X$ is efficient. The general relationship between the two concepts is summarized by Proposition 1.

Proposition 1. *For any bidder $i \in N$ and any feasible allocation $x \in X$:*

(a) $\quad y_i^Q(x) = y_i^V + \Delta v_{-i}(x);$

Table 2: *Example of the Vickrey and Quasi Vickrey outcomes*

	Bidder 1	**Bidder 2**	**Bidder 3**
Values (A, B, AB):	$v_1 = (18, 12, 40)$	$v_2 = (50, 40, 53)$	$v_3 = (3, 4, 10)$
Vickrey outcome	$x_1^* = B$ $y_1^V = 4$ $\pi_1^V = 8$	$x_2^* = A$ $y_2^V = 28$ $\pi_2^V = 22$	$x_3^* = \emptyset$ $y_3^V = 0$ $\pi_3^V = 0$
A quasi Vickrey outcome for $x = (\emptyset, A, \emptyset)$	$x_1^* = \emptyset$ $y_1^Q = 4$ $\pi_1^Q = -4$	$x_2^* = A$ $y_2^Q = 40$ $\pi_2^Q = 10$	$x_3^* = \emptyset$ $y_3^Q = 12$ $\pi_3^Q = -12$

(b) $\pi_i^Q(x) = \pi_i^V - \Delta v_N(x).$

Proposition 1 links the Vickrey outcome and all quasi Vickrey outcomes. Relative to its true Vickrey payment, the quasi Vickrey payment of bidder i is increased/decreased by the net value gain of its opponents. Payoffs of all bidders in a quasi Vickrey outcome based on allocation x are reduced relative to their Vickrey payoffs by the aggregate inefficiency of allocation x implying that a quasi Vickrey outcome does not have to be individually rational.

Before presenting our main results, we review some definitions and results related to the generalized axiom of revealed preference (GARP) in Section 2.1.

2.1 Generalized Axiom of Revealed Preference

The main contribution of this paper is heavily based on *the Generalized Axiom of Revealed Preference (GARP)*, a well-known rationality concept in economics. Here we briefly review the basic definitions and results from the GARP literature and derive some additional results. All definitions and results are stated for value functions satisfying (A1)–(A3).

At every time $t \in [0, T]$, the seller announces a price vector $p(t) = (p^1(t), ..., p^K(t)) \in \mathbb{R}_+^K$, i.e., the current prices for all K goods. The price path $p(\cdot)$ is assumed to be continuous and piecewise linear on $[0, T]$.

Given the current prices $p(t)$, each bidder i replies with a single bundle $x_i(t)$ from the set Ω. The demand function $x_i(\cdot)$ for each bidder i is assumed to be a right-continuous function that has at most a finite number of discontinuities on $[0, T]$, i.e., a piecewise constant function.[1]

Bidder i is said *to bid straightforwardly according to value function* $v(\cdot)$, if at every time $t \in [0, T]$, its chosen bundle $x_i(t)$ belongs to its demand set at prices $p(t)$, i.e.:

$$x_i(t) \;\in\; \arg\max_{z \in \Omega} \{ \, v(z) \;-\; p(t) \cdot z \, \}. \tag{2.2}$$

Bidder i is said *to bid truthfully* if she bids straightforwardly according to its true value function $v_i(\cdot)$.

Suppose that the auctioneer wants to test whether the observed demand function $x_i(t)$ of bidder i can be rationalized by some value function. The traditional verification process based on GARP (for value functions satisfying (A1)–(A3)) proceeds as follows:

(1) Select a sample size $M \geq 2$ and take any $\{t_1, ..., t_M\} \in [0, T]$;

(2) Calculate the **net payment** NP that bidder i would have agreed to make (according to her revealed preferences) if she trades bundle $x_i(t_M)$ for bundle $x_i(t_1)$, then she trades bundle $x_i(t_1)$ for bundle $x_i(t_2)$, then $x_i(t_2)$ for $x_i(t_3)$, and so on until she trades back to the original bundle $x_i(t_M)$. This net payment is calculated as follows:

$$\begin{aligned} NP \;\;=\;\; & p(t_1)[x_i(t_1) - x_i(t_M)] + p(t_2)[x_i(t_2) - x_i(t_1)] \,+\, \ldots \\ & \ldots \,+\, p(t_M)[x_i(t_M) - x_i(t_{M-1})]; \end{aligned} \tag{2.3}$$

(3) If the resulting payment is zero or negative ($NP \leq 0$), demand choices of bidder i are consistent with GARP. However, if the resulting payment is positive ($NP > 0$), bidder i's demand choices violate GARP. Violation of GARP is equivalent to existence of a "money pump".

[1] This assumption is without loss of generality. It can be shown that a piecewise constant demand function exists for any function $v(\cdot)$ satisfying assumptions (A1)–(A3) provided that the price path $p(\cdot)$ is continuous and piecewise linear on $[0, T]$.

The main result in GARP literature is the theorem due to Afriat (1967).[2] The theorem links GARP and the existence of the value function that rationalizes demand $x_i(.)$. In short, a rationalizing value function exists as long as there are no GARP violations for any choice of sample size M and $\{t_1, ..., t_M\} \in [0, T]$ in the process above. Using our notation and assumptions, the "continuous version" of the Afriat's theorem is stated as follows:

Afriat's Theorem (1967). *Given price path $p(.)$, bidder i's demand $x_i(.)$ is rationalized by some value function $v(.)$ if and only if her demand $x_i(.)$ satisfies GARP on $[0, T]$, i.e. when:*[3]

$$p(s)[x_i(s) - x_i(t)] \; + \; \int_s^t p(u)\, dx_i(u) \; \leq \; 0 \qquad \forall\, t, s \in [0, T] \tag{2.4}$$

Now we introduce a convenient notion of irrationality according to GARP. Suppose that demand $x_i(.)$ of bidder i satisfies GARP on $[0, T]$. Consider a time interval $[0, t]$ where $t \leq T$ and change bidder i's demand at time t from her actual demand $x_i(t)$ to package z. Note that demanding package z at time t can cause a GARP violation on $[0, t]$. To eliminate GARP violation, we add a large enough constant $\lambda \geq 0$ to ensure that the left-hand side of (2.4) is not positive, i.e.:

$$p(s)[x_i(s) - z] + \int_s^t p(u)\, dx_i(u) + p(t)[z - x_i(t)] - \lambda \leq 0 \qquad \forall\, s \in [0, t] \tag{2.5}$$

Intuitively, the constant λ can be interpreted as a discount applied to bundle z at time t that rationalized choosing bundle z. We define the GARP violation for bundle z at time t to be a minimum discount $\lambda \geq 0$ that is needed to rationalize bidding for bundle z at time t given a prior bidding history for bidder i. Formally, it is the minimum λ that satisfies (2.5) that can be restated as follows:

[2] Rochet (1987) provides a proof of the Afriat's Theorem for the case of quasilinear utilities.

[3] The integral term in (2.4) and all similar formulas in the paper is a Stieltjes integral. A definition of the Stieltjes integral can be found in Apostol (1957, Definition 9-1). See Ausubel (2006, pp. 690) for a short discussion.

$$gv_i(t,z) \;=\; \max_{s\in[0,t]} \left\{ p(s)[x_i(s)-z] + \int_s^t p(u)\,dx_i(u) + p(t)[z-x_i(t)] \right\} \quad (2.6)$$

By construction, $gv_i(t,z) \geq 0$ for all $z \in \Omega$ and all $t \in [0,T]$. If $gv_i(t,z) = 0$, then bidder i would not have violated GARP by bidding for bundle z at time t, and if $gv_i(t,z) > 0$, then bidding for z at time t would have violated GARP.

In the next lemma we prove important properties of the GARP violation defined in (2.6). First, we show that if bidder i has previously demanded package z, then the formula for the GARP violation can be simplified. Second, the GARP violation satisfies natural monotonicity properties when the price path $p(\cdot)$ is nondecreasing.

Lemma 1. *Suppose that GARP violation $gv_i(t,z)$ is defined by (2.6). Then*

(a) for any $z \in \Omega$ such that $z = x_i(s)$ for some $s \in [0,t]$,

$$gv_i(t,z) \;=\; \int_s^t p(u)\,dx_i(u) \;+\; p(t)\,[z \;-\; x_i(t)]\,; \quad (2.7)$$

(b) for a nondecreasing price path $p(\cdot)$ and any two packages $z, z' \in \Omega$ such that $z \;\leq\; z'$,

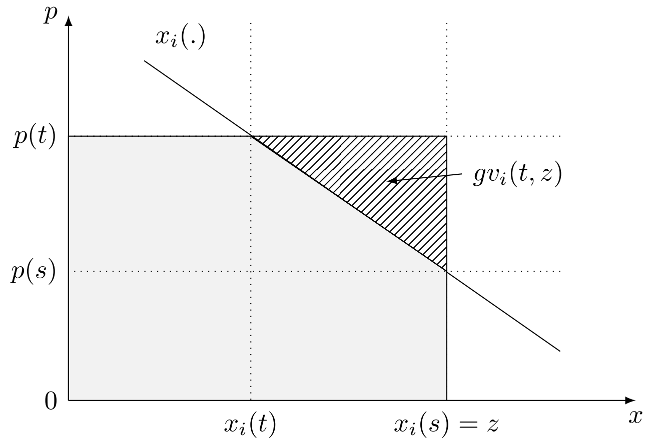
$$0 \;\leq\; gv_i(z',t) \;-\; gv_i(z,t) \;\leq\; p(t)\,[z' \;-\; z]. \quad (2.8)$$

Figure 1 illustrates the GARP violation for a setting with homogeneous good ($K = 1$). We assume that there are many identical units of the good ($S >> 1$) such that a continuous demand function can be a good visual approximation of the actual step-function demand $x_i(\cdot)$. Suppose that at time s given price of $p(s)$, bidder i bids for z units; and she bids for $x_i(t)$ units (where $x_i(t) < z$) at a later time $t > s$ given price $p(t) > p(s)$. In this case, the "crossed" area in Figure 1 corresponds to the GARP violation $gv_i(t,z)$ that bidder i would have caused by biding for z units at price $p(t)$.

Lemma 2 provides useful characterizations of the underlying value function in terms of GARP violations for a straightforward bidder.

Lemma 2. *Suppose that bidder i bids straightforwardly according to $v(\cdot)$ on $[0,T]$, then*

Figure 1: *GARP Violation in a Homogeneous Setting*



(a) *for all* $t \in [0, T]$

$$v(x_i(t)) = v(x_i(T)) - \int_t^T p(u)dx_i(u), \tag{2.9}$$

or, equivalently,

$$v(x_i(t)) = v(x_i(T)) + p(T)[x_i(t) - x_i(T)] - gv_i(T, x_i(t)) \tag{2.10}$$

(b) *for all* $z \in \Omega$

$$v(z) \leq v(x_i(T)) + \min_{s\in[0,T]} \left[p(s)[z - x_i(s)] - \int_s^T p(u)dx_i(u) \right], \tag{2.11}$$

or, equivalently,

$$v(z) \leq v(x_i(T)) + p(T)[z - x_i(T)] - gv_i(T, z). \tag{2.12}$$

Lastly, we define a notion of GARP violation for a coalition of bidders. For coalition $M \subseteq N$, *the joint GARP violation* when bidders in M jointly bid for bundle Z at time t is given by:

$$gv_M(t, Z) = \min_{\sum_M z_j = Z} \left\{ \sum_M gv_j(t, z_j) \right\} \tag{2.13}$$

Intuitively, $gv_M(t, Z)$ is the minimum total GARP violation that bidders in M can achieve when they collectively bid for bundle Z at time t.

3 Main Results

We present the main results in the following order. Results related to the primary stage and quasi Vickrey outcomes are provided in Section 3.1. Section 3.2 provides results for the secondary stage that converts the approximation of the quasi Vickrey outcome based on the final clock allocation to an approximation of the Vickrey outcome. Finally, Section 3.3 contains two detailed examples of the proposed auction design.

3.1 Primary Stage

The primary stage is implemented as a standard clock auction procedure. The auctioneer initializes K price clocks at zero. At each time $t \in [0, +\infty)$, the auctioneer announces a price vector $p(t) \in \mathbb{R}^K_+$ to bidders and each bidder i replies with bundle $x_i(t)$. Denote the aggregate demand of bidders in coalition M at time t as $x_M(t) = \sum_M x_i(t)$. The auctioneer sets the clock price for good k at time $t > 0$ as follows:

$$p^k(t) = \begin{cases} p^k(\underline{t}) + (t - \underline{t}) & \text{if } x^k_N(\underline{t}) > s^k \\ p^k(\underline{t}) & \text{if } x^k_N(\underline{t}) \leq s^k \end{cases} \tag{3.1}$$

where $\underline{t}$ is the highest integer such that $\underline{t} < t$. Intuitively, the price adjustment described in (3.1) uses the aggregate demand of bidders at integer time $\underline{t}$ to determine which clock prices to increase in the time interval $[\underline{t}, \underline{t} + 1]$ and then increases them at the constant speed. The procedure always results in a nondecreasing continuous piecewise-linear price path $p(\cdot)$.[4]

The auctioneer stops the clock procedure when the current aggregate demand is equal or below the supply for each good. Let T denote termination time, let $x(T) = (x_1(T), ..., x_n(T))$ denote the final clock allocation and let $U = S - x_N(T) \geq 0$ denote the excess supply at termination time (i.e., the undersell). At this point, the primary stage is over, and the auction pro-

[4] This price adjustment process avoids any possibility of infinite price oscillations. See Gul and Stacchetti (2000) and Ausubel (2006).

ceeds to the secondary stage where the winning allocation and payments are determined.

Before describing the secondary stage, we prove that the auctioneer can approximate the quasi Vickrey outcome $(x(T),\, y^Q(x(T)))$ with certain level of precision relying on demand information collected during the primary stage (Theorems 1 - 4).

Theorem 1. *If all bidders in N bid truthfully in the primary stage, then the primary stage ends in finite time and*

$$\Delta v_N(x(T)) \;\leq\; p(T)\,U \;-\; gv_N(T,S) \tag{3.2}$$

where $\Delta v_N(x(T))$ is the inefficiency of the final clock allocation $x(T)$. Furthermore, if there is no undersell, then $(x(T),\, y^Q(x(T)))$ is the Vickrey outcome.

Next theorem introduces *an upper exposure formula* that provides an upper bound for the quasi Vickrey payment. Intuitively, the auctioneer considers all possible value functions that could have generated the observed demands for all bidders and find the ones that generate the highest possible quasi Vickrey payment for each bidder. The theorem also establish an alternative formulation for the upper exposure using the concept of GARP violation.

Theorem 2. *If all bidders in N_{-i} bid truthfully in the primary stage, then*

$$y_i^Q(x(T)) \;\leq\; UE_i(T)$$

where $UE_i(T)$ is the upper exposure of bidder i given by

$$UE_i(T) \;=\; \max_{\sum_{j\neq i} z_j = S} \left[\sum_{j\neq i} \min_{t_j\in[0,T]} \left\{ p(t_j)[z_j - x_j(t_j)] \;-\; \int_{t_j}^{T} p(u)dx_j(u) \right\} \right]. \tag{3.3}$$

The upper exposure of bidder i can be alternatively stated as:

$$UE_i(T) \;=\; p(T)\,[x_i(T) \;+\; U] \;-\; gv_{-i}(T,S) \tag{3.4}$$

For intuition, consider the upper exposure formula (3.4) and suppose that $U = 0$. In this case, the upper exposure formula approximates the true

Vickrey payment. The upper exposure of bidder i is the nominal price of the winning bundle $x_i(T)$ at time T minus a discount—a hypothetical GARP violation for coalition N_{-i} when they jointly bid for the full supply vector S at time T. If the joint GARP violation is zero, then, bidders in N_{-i} could potentially have the same marginal value for the items in $x_i(T)$ as bidder i, and therefore bidder i's opportunity costs can be as high as the full nominal price of the winning bundle. However, if the joint GARP violation is positive, bidder i has a strictly higher value for items in $x_i(T)$ than its opponents, and therefore deserves a discount on the nominal price of bundle $x_i(T)$.

We now turn to derivation of the lower bound. To simplify notation, we define a set of times where aggregate demand of bidders in coalition M (possibly, at different times) is less than supply:

$$T_M = \left\{ \{t_j\}_{j \in M} : \quad t_j \in [0, T] \quad \forall j \in M \quad \text{and} \quad \sum_{j \in M} x_j(t_j) \leq S \right\} \tag{3.5}$$

In Theorem 3, we derive *a lower exposure formula* that bounds quasi Vickrey payments from below. The lower exposure formulas use revealed demand to find the highest confirmed value of opponents for goods demanded by bidder i. The theorem also establish an alternative formulation for the lower exposure using the concept of GARP violation.

Theorem 3. *If all bidders in N_{-i} bid truthfully in the primary stage, then*

$$LE_i(T) \;\leq\; y_i^Q(x_i(T)) \tag{3.6}$$

where $LE_i(T)$ is the lower exposure of bidder i given by

$$LE_i(T) \;=\; \max_{\{t_j\} \in T_{-i}} \left[-\sum_{j \neq i} \int_{t_j}^{T} p(u) dx_j(u) \right], \tag{3.7}$$

The lower exposure of bidder i can be alternatively stated as:

$$LE_i(T) \;=\; p(T)\,[x_i(T) + U] \;-\; \left(\sum_{j \neq i} gv_j(T, x_j(\hat{t}_j)) \;+\; p(T) \left[S - \sum_{j \neq i} x_j(\hat{t}_j) \right] \right) \tag{3.8}$$

where $\{\hat{t}_j\} \in T_{-i}$ solves (3.7).

Next we derive an upper bound on the approximation errors for exposure formulas. Consider an economy that consists of bidders in coalition $M \subseteq N$ and allocation z such that $\sum_M z_j = S$. If bidders were truthful during the primary stage, then using revealed preferences at times $\{t_j\}_{j \in M}$ such that $t_j \in [0, T]$ for all $j \in M$ we have:

$$\sum_{j \in M} v_j(z_j) \ - \ \sum_{j \in M} v_j(x_j(t_j)) \ \ \leq \ \ \sum_{j \in M} p(t_j)[z_j - x_j(t_j)] \tag{3.9}$$

If there exits $\{t_j\} \in T_M$ such that the right-hand side of (3.9) is negative, then allocation z must be inefficient in this economy since it is dominated by a feasible allocation $\{x_j(t_j)\}_{j \in M}$. For other allocations, the right-hand side of (3.9) bounds the maximum value gain of allocation z over $\{x_j(t_j)\}_{j \in M}$. We define a measure of maximum efficiency gain for economy consisting of bidders in M as follows:

$$\Psi_M \ = \max_{\sum\limits_{j \in M} z_j = S} \ \ \min_{\{t_j\} \in T_M} \left[\sum_{j \in M} p(t_j) \left[z_j - x_j(t_j) \right] \right]. \tag{3.10}$$

Intuitively, this measure provides an upper bound on the additional value for this economy that was not revealed by the primary stage. By construction, Ψ_M must be nonnegative. The next lemma shows that Ψ_M can be bounded from above by a minimized price measure for excess supply.

Lemma 3. *For any coalition $M \subseteq N$*

$$\Psi_M \ \leq \ \min_{t:\, x_M(t) \leq S} \ p(t) \left[S - x_M(t) \right]. \tag{3.11}$$

It follows that if at some point during the primary stage $x_M(t) = S$, then $\Psi_M = 0$ and there is no hidden value left in the economy consisting of bidders in M. This simply means that the efficient allocation has been identified which we already know due to the first welfare theorem.

Theorem 4 proves that Ψ_{-i} provides an upper bound for the approximation error for both exposure formulas and Corollary 1 specifies two sufficient conditions for an error-free approximation.

Theorem 4. *If all bidders in N_{-i} bid truthfully in the primary stage, then*

$$UE_i(T) \; - \; LE_i(T) \; \leq \; \Psi_{-i}. \tag{3.12}$$

Corollary 1.

(a) Suppose that all bidders in N_{-i} bid truthfully in the primary stage. If there exists $t \in [0, T]$ such that $x_{-i}(t) = S$, then

$$LE_i(T) \; = \; y_i^Q(x_i(T)) \; = \; UE_i(T).$$

(b) If there are only two bidders and both of them bid truthfully in the primary stage, then

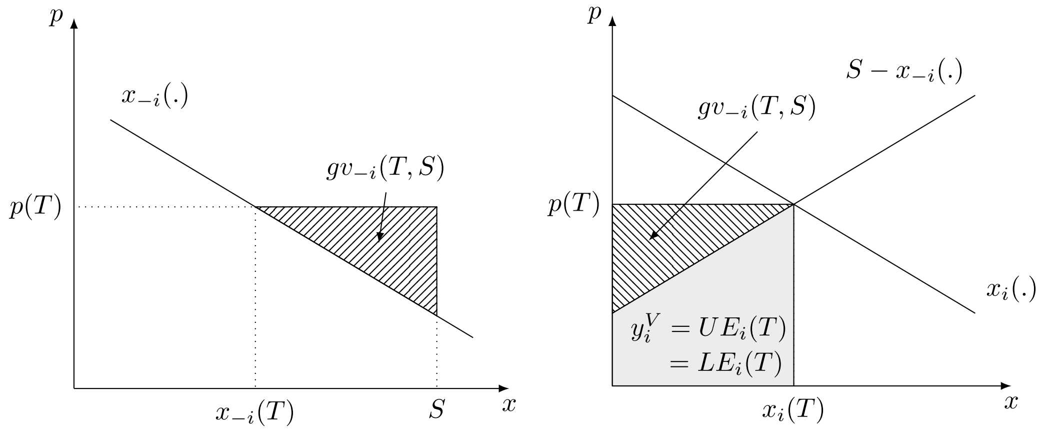
$$LE_i(T) \; = \; y_i^Q(x_i(T)) \; = \; UE_i(T) \qquad i \; = \; 1, 2.$$

Figure 2 illustrates the GARP violation for a setting with homogeneous good ($K = 1$). We assume that there are many identical units of the good ($S >> 1$) such that continuous demand functions is a good visual approximation of the actual step-function demands. Assuming that all bidders reduce their demand in 1-unit decrements implies a zero undersell $U = 0$ and an error-free approximation $\Psi_{-i} = 0$, which in turn implies implementing the true Vickrey outcome. Panel A shows (crossed area) the GARP violation for coalition N_{-i} at termination time for the full supply S. The same area is depicted in Panel B corresponding to a standard Vickrey discount to the uniform payment $p(T)\, x_i(T)$ of bidder i.

All results in this section are derived assuming straightforward bidding. To ensure that the exposure formulas continue to work as intended, the auctioneer must employ an activity rule based on GARP to ensure that each demand can be rationalized by a value function at all times.

Activity Rule for the Primary Stage: Bidder i is allowed to bid for bundle $x \in \Omega$ at time $t > 0$ if and only if $gv_i(t, x) = 0$. In other words, bidding for bundle x at time t does not cause GARP violation given the prior bidding history.

Figure 2: *GARP Violation and Exposure*



Panel A: GARP Violation **Panel B:** Exposure Pricing

If the primary stage ends with zero excess supply, the final clock allocation is efficient and exposure formulas approximate the true Vickrey payments. It is natural for the auctioneer to charge bidder i a payment of $E_i(T)$ where

$$LE_i(T) \leq E_i(T) \leq UE_i(T), \tag{3.13}$$

but the auctioneer may also charge an amount outside of this range for practical reasons (see Section 5.1 and Appendix A). For the rest of the paper, we refer to any calculation that produces $E_i(T)$ in the range:

$$0 \leq E_i(T) \leq p(T)\,[x_i(T) + U] \tag{3.14}$$

as exposure formula. Note that any exposure formula that satisfies (3.14) is individually rational for truthful bidders when $U = 0$.

With zero excess supply, the auctioneer can simply award the final clock bundles to bidders and charge them according to some exposure formula without going through the secondary stage. However, with positive undersell, the secondary stage is needed to improve the efficiency of the allocation and to adjust exposure payments.

3.2 Secondary Stage

The primary stage described in the previous section produces an approximation of a quasi Vickrey outcome based on the final clock allocation $x(T)$. The purpose of the secondary stage is to replace the approximation of this quasi Vickrey outcome with an approximation of the true Vickrey outcome using the link established by Proposition 1.

The secondary stage works as a sealed-bid package auction. Denote $b_i(z)$ a bid of bidder i for bundle z made (or imputed by the auctioneer) in the secondary stage. For all bidders, the bid for the null bundle is restricted to be zero, i.e. $b_i(0) = 0$ for all $i \in N$. All bids placed in the secondary stage have to be linked to bids made in the primary stage via an appropriate activity rule; otherwise, the primary stage is irrelevant. For our purposes, we adopt the following activity rule:

Activity Rule for the Secondary Stage: There exists a constant $c_i \geq 0$ for bidder i such that $b_i(x_i(T)) \; = \; p(T)\, x_i(T) + c_i$ (where $c_i = 0$ if $x_i(T) = 0$) and for all other bundles $z \in \Omega$

$$0 \;\leq\; b_i(z) \;\leq\; p(T)\, z \;+\; c_i \;-\; gv_i(T, z). \tag{3.15}$$

It is convenient to restate the upper limit of (3.15) as follows:

$$b_i(z) \;-\; b_i(x_i(T)) \;\leq\; \; p(T)\, [z \;-\; x_i(T)] \;-\; gv_i(T, z). \tag{3.16}$$

The activity rule serves two purposes. First, it ensures that bidder i bids at least the final nominal price for the final clock bundle $x_i(T)$. Second, for any other bundle z, the activity rule uses Lemma 2(b) to derive the highest possible value that is consistent with GARP and uses it as the upper limit for $b_i(z)$. Note that this activity rule does not enforce the lower bound on bids for bundles demanded at any time $t < T$.[5]

[5]In a typical CCA, all clock bids are "live" in the supplementary stage creating the lower bound for the bid amount for any bundle that received bid during the clock stage. We do not enforce such lower bounds for the ease of exposition. Treating all clock bids as "live" does not affect our results.

Given all bids placed in the secondary stage, the auctioneer solves for a feasible allocation $x = (x_1, ..., x_n)$ that maximizes $\sum_N b_j(x_j)$. The activity rule (3.15) can significantly restrict the set of possible winning allocations, so there can be a lot of bids that would never win. A bid $b_i(z)$ is said to be *a potentially winning bid* for bidder i if there exist allocation allocation $(x_1, ..., x_n) \in X$ where $x_i = z$ and a set of bids $\{b_j(x_j)\}_{j \in N}$ that satisfies the activity rule (3.15) such that

$$\sum_N b_j(x_j) \;\geq\; \sum_N b_j(x_j(T)). \tag{3.17}$$

Obviously, bid $b_i(x_i(T))$ is a potentially winning bid for each bidder i. Bid $b_i(z)$ for bundle z is said to be *a losing bid* when it is not a potentially winning bid.

Proposition 2. *Suppose that all bids placed in the secondary stage are constrained by the activity rule (3.15). Bid $b_i(z)$ is a potentially winning bid for bidder i if and only if*

$$b_i(z) \,-\, b_i(x_i(T)) \;\geq\; p(T)\,[z \,-\, x_i(T)] \,-\, \big[p(T)\,U \,-\, gv_{-i}(T, S-z)\big]. \tag{3.18}$$

It follows that any bid $b_i(z)$ for bundle z that satisfies the activity rule is a losing bid if

$$gv_i(T, z) \;+\; gv_{-i}(T, S-z) \;>\; p(T)\,U. \tag{3.19}$$

Inequality (3.19) is intuitive. When $p(T)\,U$ is relatively small, essentially all bundles in Ω become losing for each bidder i since both GARP violation terms on the left hand-side of (3.19) are nonnegative by construction and considerable deviation from the final clock allocation $x(T)$ would cause at least one of them to become positive. When $p(T)\,U = 0$, all bids placed in the secondary stage are in fact losing bids (with the exception of possibility for multiple efficient allocations).

The main purpose of the primary stage is to simplify (and sometimes eliminate) bidding in the secondary stage and provide bidders with some form of "privacy preservation". As such, fully truthful bidding in the secondary stage in the sense of $b_i(z) = v_i(z)$ for all $z \in \Omega$ and all $i \in N$ is not needed to reach the efficient allocation. Plugging $z = 0$ into (3.18) shows that the null bid is a potentially winning bid for bidder i only when $b_i(x_i(T)) \leq UE_i(T)$. But then bidder i is guaranteed to win some goods after the secondary stage when she bids more than her upper exposure for the final clock bundle $x_i(T)$. Intuitively, the upper exposure provides bidders with the minimum absolute level for their bids to ensure winning. Without loss of generality, we assume that the auctioneer breaks any possible ties when maximizing $\sum_N b_j(x_j)$ such that $x_i \neq 0$ for each bidder i who bids $b_i(x_i(T)) = UE_i(T)$.

For each bidder i, denote the true marginal value between bundles $z \in \Omega$ and $x_i(T)$ as

$$mv_i(z) \;=\; v_i(z) \;-\; v_i(x_i(T)). \tag{3.20}$$

Next, we define the appropriate notion of truthful bidding for the secondary stage. It is said that bidder i bids *semi-truthfully* in the secondary stage when the following two conditions are true:

1. The bid $b_i(x_i(T))$ for the final clock bundle $x_i(T)$:

   - if the upper exposure $UE_i(T)$ of bidder i exceeds the true value for bundle $x_i(T)$, then bidder i bids its true value for $x_i(T)$, i.e.:

     $$UE_i(T) \;\geq\; v_i(x_i(T)) \qquad \Rightarrow \qquad b_i(x_i(T)) \;=\; v_i(x_i(T));$$

   - if the upper exposure $UE_i(T)$ of bidder i is lower than the true value for bundle $x_i(T)$, then bidder i bids any amount above the exposure level, i.e.:

     $$UE_i(T) \;<\; v_i(x_i(T)) \qquad \Rightarrow \qquad b_i(x_i(T)) \;\geq\; UE_i(T).$$

2. Bids for any non-null bundle $z \neq x_i(T)$:

   - if a bid

     $$b_i(z) \;=\; b_i(x_i(T)) \;+\; mv_i(z)$$

     is a potentially winning bid, then bidder i places this bid. If this bid is a losing bid, then bidder i places any losing bid for bundle z that is consistent with the activity rule (3.15).

It is easy to verify that semi-truthful bidding in the secondary stage is permitted by the activity rule provided bidder i was bidding truthfully in the primary stage.

Next we show that the auctioneer can recover the efficient allocation by simply maximizing $\sum_N b_j(x_j)$ on X provided that all bidders bid truthfully in the primary stage and semi-truthfully in the secondary stage. Let $x^* = (x_1^*, ..., x_n^*) \in X$ be the allocation that maximizes $\sum_N b_j(x_j)$. Denote $\Delta b_i(z) = b_i(x_i^*) - b_i(z)$ the implied net value gain of bidder i from bundle x_i^* relative to bundle z; and $\Delta b_M(x) = \sum_M \Delta b_j(x_j)$ the implied net value gain of coalition M from allocation x^* relative to allocation x.

Theorem 5. *Suppose that all bidders bid truthfully in the primary stage and semi-truthfully in the secondary stage. Then allocation x^* is efficient and*

$$y_i^V \;=\; y_i^Q(x(T)) \;-\; \Delta b_{-i}(x(T)) \tag{3.21}$$

Theorem 5 shows that the Vickrey payment y_i^V is directly linked to the quasi Vickrey payment $y_i^Q(x(T))$ via (3.21). It follows that Vickrey payments can be approximated by various exposure formulas developed in Section 3.1 and suggests the following payment formula for the secondary stage:

$$\tilde{y}_i \;=\; E_i(T) \;-\; \Delta b_{-i}(x(T)) \tag{3.22}$$

where $E_i(T)$ is the exposure of bidder i.

It is instructive to compare formula (3.22) to the standard Vickrey calculation. The formula consists of exposure term $E_i(T)$ and adjustment term $\Delta b_{-i}(x(T))$. Similar to the Vickrey formula, the exposure term, $E_i(T)$, is not affected by bidder i's own bids placed in the secondary stage, and the adjustment term, $\Delta b_{-i}(x(T))$, is only affected by bidder i's bids through the winning allocation. But there is also a notable difference between two formulas. The exposure term $E_i(T)$ is not unaffected by any bids submitted in the secondary stage. As a result, payment $\tilde{y}_i$ is not guaranteed to be in the $[0,\, b_i(x_i^*)]$ interval. To make sure that final payments are nonnegative and individually rational, we cap $\tilde{y}_i$ from below by 0 and from above by the winning bid amount $b_i(x_i^*)$ as follows:

$$y_i^* = \begin{cases} 0 & \text{if} \quad \tilde{y}_i < 0 \\ \tilde{y}_i & \text{if} \quad 0 \leq \tilde{y}_i \leq b_i(x_i^*) \\ b_i(x_i^*) & \text{if} \quad \tilde{y}_i > b_i(x_i^*) \end{cases} \tag{3.23}$$

We finish this section by proposing an auction design that is strongly related to the popular CCA format. This auction design is best described as an improved version of the CCA format — an Enhanced Combinatorial Clock Auction (ECCA).

> **ECCA:** The primary stage follows Section 3.1. The secondary stage follows Section 3.2. The winning allocation $x^* = (x_1^*, ..., x_n^*)$ is determined by maximizing $\sum_N b_j(x_j)$ such that $x \in X$ and bidder payments $y^* = (y_1^*, ..., y_n^*)$ are determined by (3.23).

The first desirable property of the ECCA is that there is no need to hold the secondary stage when the primary stage ended with no excess supply.

Proposition 3. *If there is no excess supply at the end of the primary stage, i.e., $U = 0$, the ECCA outcome is given by $x_i^* = x_i(T)$ and $y_i^* = E_i(T)$. In other words, all bids collected in the secondary stage are irrelevant for determining the auction outcome.*

Proof. The activity rule (3.15) ensures that the final clock allocation will maximize the objective $\sum_N b_j(x_j)$. But then $\Delta b_{-i}(x(T)) = 0$ for all $i \in N$ and $y_i^* = \tilde{y}_i = E_i(T)$ as long as $E_i(T)$ satisfies (3.14). □

Before discussing other properties and incentives that bidders face in the ECCA, we provide two detailed examples.

3.3 Examples of ECCA

To make the following examples concrete, we provide exposure calculations using both the upper and lower exposure formulas.

In the first example, consider the setting with two unique items, A and B, and two bidders that has been used in Section 1. Table 3 provides bidder values and truthful demands for each bidder and the corresponding price path. At the end of the primary stage with clock prices reaching $p(T) = (30, 20)$, Bidder 1 demands AB and Bidder 2 demands $\emptyset$ resulting in zero excess supply. Since the primary stage ended with zero excess supply, the secondary stage is redundant. The final clock allocation is efficient and both exposure calculations return the true Vickrey payment (Corollary 1(b) applies to settings with two bidders).

Table 3: *Example 1*

	Bidder 1	**Bidder 2**
Values:	$v_1(A) = 20$	$v_2(A) = 30$
	$v_1(B) = 10$	$v_2(B) = 20$
	$v_1(AB) = 60$	$v_2(AB) = 33$
	Primary Stage	
Clock Prices (p_A, p_B)	Clock Demands	
(0, 0) → (3, 3)	AB	AB
(3, 3) → (13, 3)	AB	A
(13, 3) → (30, 20)	AB	A or B*
$p(T) = (30, 20)$	AB	$\emptyset$
	Exposure Calculation	
Lowe Exposure $LE_i(T)$	33	0
Upper Exposure $UE_i(T)$	33	0
	Auction Outcome	
Winning allocation	AB	$\emptyset$
Payment y_i^* based on $UE_i(T)$	33	0

* – Bidder 2 switches between A and B while prices rise from (13, 3) to (30, 20).

For the second example, consider the setting with two unique items, A and B, and three bidders that has been used in Section 2. Table 4 provides bidder values and truthful demands for each bidder and the corresponding price path. At the end of the primary stage with clock prices reaching $p(T) = (25, 15)$, Bidders 1, 2 and 3 demand ($\emptyset$, A, $\emptyset$) correspondingly resulting in excess supply.[6]

[6] We assume that Bidder 2 demands item A when indifferent between items A and B.

Table 4: *Example 2*

	Bidder 1	**Bidder 2**	**Bidder 3**
Values:	$v_1(A) = 18$ $v_1(B) = 12$ $v_1(AB) = 40$	$v_2(A) = 50$ $v_2(B) = 40$ $v_2(AB) = 53$	$v_3(A) = 3$ $v_3(B) = 4$ $v_3(AB) = 10$
Primary Stage			
Clock Prices (p_A, p_B)	Clock Demands		
(0, 0) → (3, 3)	AB	AB	AB
(3, 3) → (5, 5)	AB	A	AB
(5, 5) → (15, 5)	AB	A	$\emptyset$
(15, 5) → (25, 15)	AB	A or B*	$\emptyset$
$p(T) = (25, 15)$	$\emptyset$	A	$\emptyset$
Exposure Calculation			
$LE_i(T) \leq y_i^Q(x(T)) \leq UE_i(T)$	$3 \leq 4 \leq 5$	$40 \leq 40 \leq 40$	$3 \leq 12 \leq 15$
Approximation Error Ψ_{-i}	2	0	15
Secondary Stage			
Losing Bids: $c_1 = 0$, $c_2 = 15$, $c_3 = 0$	$b_1(A) < 10$ $b_1(B) < 0$ $b_1(AB) < 25$	$b_2(B) < 15$ $b_2(AB) < 40$	$b_3(A) < 10$ $b_3(B) < 0$ $b_3(AB) < 25$
Bids in Secondary Stage:	$b_1(A) = 18$ $b_1(B) = 12$ $b_1(AB) = 40$	$b_2(A) = 40$ $b_2(B) = 30$ $b_2(AB) = 43$	$b_3(B) = 4$
Auction Outcome			
Winning allocation	B	A	$\emptyset$
Adjustment $\Delta b_{-i}(x(T))$	0	12	12
Payment $\tilde{y}_i$ based on $LE_i(T)$	3	28	-9
Payment $\tilde{y}_i$ based on $UE_i(T)$	5	28	3
Payment y_i^* based on $LE_i(T)$	3	28	0
Payment y_i^* based on $UE_i(T)$	5	28	0

* – Bidder 2 switches between A and B while prices rise from (15, 5) to (25, 15).

For the same set of values, we have already calculated the Vickrey outcome and a quasi Vickrey outcome based on the allocation ($\emptyset$, A, $\emptyset$) in Section 3 (see

Table 2). The exposure calculations in Table 4 approximate the quasi Vickrey payment $y^Q = (4, 40, 12)$. Note that both exposure calculations are error-free for Bidder 2 ($\Psi_{-2} = 0$ due to Corollary 1(a)); relatively precise for Bidder 1 with $\Psi_{-1} = 2$; and rather imprecise for Bidder 3 with $\Psi_{-3} = 15$.

For the secondary stage, Table 4 reports losing bid thresholds and one possible set of semi-truthful bids. For Bidder 1, bidding the true values is the only bid profile that is consistent with semi-truthful bidding. Bidder 2 raises its bid for item A to 40 (her exposure amount) and bid true relative values for item B and package AB. Bidder 3 has to bid truthfully for item B but any bids that can be placed for item A or package AB will be losing bids (so no need to place them).

Given the bids placed in the secondary stage, the winning allocation is $(B, A, \emptyset)$. The final payments y_i^* are calculated using formula (3.23). Bidder 1 pays either 3 or 5 depending on which exposure formula is used. Note that her true Vickrey payment is 4. Both Bidder 2 and Bidder 3 pay their Vickrey payments. However, there is a difference between payment calculations for Bidder 2 and Bidder 3. For Bidder 2, her Vickrey payment of 28 is fully recovered since there was no approximation error in her exposure calculation in the primary stage. In contrast, the Vickrey payment for Bidder 3 is recovered simply by luck because Bidder 3 does not win anything (and then her Vickrey payment should be zero). If instead the final payments for $\tilde{y}_i$ were calculated using formula (3.22), the final payment of Bidder 3 would have been either -9 or 3.

In the next section, we provide results on bidder incentives for truthful bidding during both stages of the ECCA.

4 Bidder Incentives

The ECCA format approximates the Vickrey outcome instead of implementing it. Since payments can differ from the Vickrey payments, the full equilibrium analysis in a general private value setting is too complex. To the best of our knowledge, none of the auction designs that are used in modern spectrum auctions are fully incentive compatible.

Given the two stages of the ECCA, it is natural to start the analysis with bidding incentives in the secondary stage. We have already highlighted the similarities between the standard Vickrey calculation and the formula (3.22) for calculating $\tilde{y}_i$ that in turn is used as a main ingredient for calculating y_i^*. First, we show that these similarities across two formulas immediately lead to the well-known incentive property.

Proposition 4. *If the payment of bidder i is determined by $\tilde{y}_i$, then bidding semi-truthfully in the secondary stage (if allowed by the activity rule (3.15)) is a weakly dominant strategy for bidder i.*[7]

Given that the actual payment y_i^* is equal to $\tilde{y}_i$ in many circumstances, incentives for semi-truthful bidding in the secondary stage can be strong. At the same time, when bidder i expects its $\tilde{y}_i$ to be capped or can bid in a way that would cap its $\tilde{y}_i$, deviating from semi-truthful bidding can be profitable. In the next lemma, we specify general conditions under which bidder i's final payment $y_i^* = \tilde{y}_i$.

Lemma 4. *Suppose that all bids placed in the secondary stage are constrained by the activity rule (3.15).*

(a) *If* $E_i(T) \geq UE_i(T)$, *then* $\tilde{y}_i \geq 0$;

(b) *If* $E_i(T) \leq b_i(x_i(T))$, *then* $\tilde{y}_i \leq b_i(x_i^*)$.

According to Lemma 4, when the exposure of bidder i is at least its upper exposure and bidder i bids its exposure or above on the final clock package, the payment $y_i^* = \tilde{y}_i$. Then it follows that bidding semi-truthfully is a weakly dominant strategy for bidder i if $E_i(T) \geq UE_i(T)$ and she is limited to bid such that $E_i(T) \leq b_i(x_i(T))$ by the activity rule.

[7] We use the *weakly dominant* term with a slight abuse of the common definition since in some scenarios bidders might have multiple strategies that are consistent with semi-truthful bidding.

Proposition 5. *Bidding semi-truthfully in the secondary stage of the ECCA is a weakly dominant strategy for bidder i (when allowed by the activity rule (3.15)) if*

$$UE_i(T) \ \leq\ E_i(T) \ \leq\ p(T)\, x_i(T). \tag{4.1}$$

Even when Proposition 5 dos not apply, a bidder might have strong incentives to bid semi-truthfully in a practical setting. Note that when $E_i(T) = UE_i(T)$, bidding semi-truthfully is a weakly dominant strategy on a subset of strategies that guarantee positive winnings for bidder i (recall that bidder i can ensure positive winnings by bidding $b_i(x_i(T)) > UE_i(T)$). In other words, guaranteeing positive winnings is equivalent to not being able to exploit the capping feature in the formula for calculating y_i^*.

In general, when condition (4.1) is not satisfied, bidder i can benefit from deviating in certain situations. However, gains from such deviations can be nonexistent when her exposure is not overestimated and all her opponents bid semi-truthfully. In the next theorem, we show that semi-truthful bidding by all bidders forms an ex post equilibrium of the secondary stage of the ECCA under the *"no approximation error"* condition.

Theorem 6. *Suppose that all bidders bid truthfully in the primary stage. Semi-truthful bidding by all bidders is an ex post equilibrium of the secondary stage of the ECCA if for each bidder $i \in N$, either*

$$E_i(T) \ =\ y_i^Q(x(T)); \tag{4.2}$$

or

$$y_i^Q(x(T)) \ <\ E_i(T) \ \leq\ p(T)\, x_i(T). \tag{4.3}$$

Intuition behind Theorem 6 is as follows. When all opponents of bidder i bid semi-truthfully and her exposure after the primary stage is correct, her payment is given by $\tilde{y}_i$ (i.e., it is never capped by either 0 or by $b_i(x_i^*)$) for any bid profile that she can submit. But then her best response is to bid semi-truthfully by Proposition 4. In addition, if bidder i's exposure is overestimated, semi-truthful bidding is still a best response (in fact, a weakly dominant strategy) when bidder i cannot bid less than her exposure on the final clock package due to the activity rule restriction.

Now we turn to the equilibrium analysis of the primary stage. We say that bidder i *bids truthfully in the ECCA* if she bids truthfully in the primary stage and semi-truthfully in the secondary stage. For the incentive analysis of the primary stage, it is useful to rank different exposure formulas using the following definition.

We say that *exposure formula $E(.)$ is at least (at most) as aggressive as another exposure formula $E'(.)$* if $E_i(T) \geq E'_i(T)$ ($E_i(T) \leq E'_i(T)$) for each bidder $i \in N$ and any clock price path $p(.)$, clock demands $\{x_j(.)\}_{j=1}^n$ and the corresponding termination time T.

The *"no approximation error"* condition (4.2) introduced above is important to rule out profitable deviations in the secondary stage. It turns out that it is also a sufficient condition for existence of the truthful equilibrium in the ECCA that employs a relatively aggressive exposure formula.

Theorem 7. *Suppose that the auctioneer uses an exposure formula that is at least as aggressive as the upper exposure formula and $E_i(T) = y_i^Q(x(T))$ for each bidder $i \in N$ given truthful bidding of all bidders in the primary stage. Then truthful bidding in the ECCA by all bidders forms an ex post equilibrium.*

When there are only two bidders, Corollary 1 showed that both the lower and the upper exposure formulas calculate the quasi Vickrey payments correctly. It follows that the ECCA with two bidders has a truthful equilibrium.

Corollary 2. *If there are only two bidders in the ECCA that uses the exposure formula that is at least as aggressive as the lower exposure formula, then truthful bidding by both bidders forms an ex post equilibrium of the ECCA.*

It is easy to see why the ECCA cannot be incentive-compatible with more than two bidders. Consider a standard Local-Local-Global (LLG) setting with two items, A and B, and three single-minded bidders with the following values: $v_1(A) = 6$, $v_2(B) = 8$ and $v_3(AB) = 10$. The Vickrey outcome for this economy consists of allocation $x^* = (A, B, \emptyset)$ and payment vector $y^V = (2, 4, 0)$.

Now suppose that the auctioneer employs the ECCA with the upper exposure formula and increases the clock price for items with excess demand with the same speed. If all bidder are truthful, the primary stage of the ECCA would terminate with the efficient allocation and a payment vector $y^* = (5, 5, 0)$. While both payments are in excess of the true Vickrey payments, they do correctly reflect the opportunity costs of Bidder 3 who offered to pay 10 for the bundle AB. To put it differently, the ECCA payments y^* belong to the core of this economy while Vickrey payments y^V are too low. Given that the ECCA design produces a core payment vector for this LLG economy, it is not at all surprising that local bidders have incentives to deviate from truthful bidding as they do in standard core-selecting auctions. At the same time, this example demonstrates that the overestimation caused by the upper exposure formula can sometimes be desirable.

Results in this section indicate that the auctioneer should prefer relatively aggressive exposure formulas such that bidder's exposure $E_i(T)$ is at least as large as bidder's quasi Vickrey payment $y_i^Q(x(T))$ (assuming truthful bidding by bidder i's opponents). The least aggressive exposure formula that always guarantee this property is the upper exposure $UE_i(T)$. The rationale for favoring the upper exposure is clear. With the upper exposure, a deviating bidder can improve its payoff by reducing the positive approximation error. When there is no approximation error, the bidder cannot gain. With a less aggressive exposure formula, such as the lower exposure, a deviating bidder might be able to drive $E_i(T)$ below $y_i^Q(x(T))$. To put it differently, under the upper exposure, bidders' incentives are directed towards driving the final outcome closer to the Vickrey outcome. In contrast, under the lower exposure, bidders' incentives are sometimes directed towards driving the final outcome away from the Vickrey outcome.

Our analysis show that in the general setting with more than two bidders, the ECCA design is not fully incentive compatible. However, our results point to a truthful bidding as a solid strategy choice in realistic settings of spectrum auctions. First, semi-truthful bidding in the secondary stage is often the best response or even a weakly dominant strategy, so bidders need to make sure that this strategy is not ruled out by the activity rule in case the secondary stage is needed. Second, profitable deviations from semi-truthful bidding when other bidder bids semi-truthfully require the knowledge of the approximation error which is unobservable. To gain, the bidder must place bids that are high enough to win but low enough to cap his payment. In case the bidder makes a mistake, the bidder gets nothing. Finally, when the upper exposure formula is used, it is impossible for a bidder to simultaneously gain from a deviation and have protection against losing.

Deriving an optimal deviation in the primary stage is not trivial. The exposure calculation for bidder i depends only on clock bids made by her opponents, so the bidder has only limited possibilities to affect it via her clock bids by manipulating the price path $p(.)$ that is governed by the aggregate demand $x_N(.)$. Affecting the price trajectory might require substantial deviations from truthful bidding during the primary stage that might prevent the bidder from being able to bid truthfully later in the primary stage or the secondary stage. In addition, the clock price incrementing policy used by the auctioneer can significantly limit the scope for such manipulations.

5 Practical Implementation

In this section we discuss a few topics related to implementing the ECCA. First, we show how the proposed design can be converted into an iterative first-price auction where bidders are dynamically informed about their prospective payment in the auction and alternative exposure formulas that are based on the aggregate demand. Second, we derive a lower bound on ECCA auction revenue at the end of the primary stage. Finally, we conclude this section by discussing alternative exposure-based auction designs that are closely related to the most popular auction designs used for spectrum auctions.

For the purposes of this section, it is convenient to restate the exposure formula in terms of its corresponding discount $d_i(T)$ such that

$$E_i(T) \;=\; p(T)\,[x_i(T) + U] \;-\; d_i(T) \tag{5.1}$$

The discount term for the upper exposure $UE_i(T)$ and the lower exposure $LE_i(T)$ are specified in Theorems 2 and 3.[8]

5.1 Iterative First-Price Implementation

The central idea behind the iterative first-pricing is to dynamically inform bidders about the current level of their payments. In the ECCA, this can be accomplished by disclosing bidder's current exposure to the bidder. Formally, the auctioneer treats each time t as if it was the termination time T for the auction. According to (5.1), the exposure of bidder i consists of three components: (1) the current clock price of the bidder i's current demand $x_i(t)$; (2) the current clock price of the undersell U; and (3) a discount term $d_i(t)$.

Suppose that the discount term $d_i(t)$ in (5.1) does not depend on demands of bidders in N_{-i} at time t. Indeed, this is true for discounts in both the upper and lower exposure formulas. Then such discount term can be calculated and communicated to bidder i simultaneously with announcing the vector of current prices $p(t)$.

Now suppose that at time t, each bidder i is informed about the current clock prices $p(t)$ and its current exposure discount $d_i(t)$. If at time t, bidder i demands bundle $x_i(t)$ and the primary stage closes with zero excess supply, then bidder i wins bundle $x_i(t)$ and pays

$$p(t)\,x_i(t) \;-\; d_i(t). \tag{5.2}$$

In this scenario, bidder i is perfectly informed about her final payment for any bundle that she could possibly demand. In case the auction closes at time t with positive excess supply, the secondary stage would be needed to produce the final allocation and payments. In this scenario, the exposure of bidder i is increased by $p(t)\,U$ to yield

$$p(t)\,x_i(t) \;+\; p(t)\,U \;-\; d_i(t), \tag{5.3}$$

[8]For example, for the upper exposure $UE_i(T)$, the discount for bidder i is given by the joint GARP violation of bidders in N_{-i} when bidding for supply S at time T, i.e., $d_i(T) = gv_{-i}(T, S)$.

and her final payment will be her exposure $E_i(T)$ adjusted to reflect the results of the secondary stage. The amount of excess supply cannot be perfectly predicted since the activity rules for the primary stage does not put limits on reducing demand. However, in practice, the amount of potential undersell can be reasonably forecasted if the aggregate demand is disclosed to bidders at each t. In either scenario, knowing the amount of discount $d_i(t)$ provides bidder i with an actionable way to calculate its final payment with some certainty.

When bidders are dynamically informed about their exposure but not about individual demands of their opponents, bidders can use their exposure for additional inference about individual demands of competitors. In such circumstances, the auctioneer may consider utilizing the exposure formulas based on the aggregate demand. We develop the exposure formulas based on aggregate demand and the associated theory in Appendix A. These exposure formulas produce less precise approximations of the quasi Vickrey payments but they only rely on aggregated information.

5.2 Auction Revenue

One of the well-known drawbacks of the CCA design is the auctioneer's inability to predict the auction revenue after the clock stage. In fact, it is possible that the bidders' payments after the supplementary round will be determined to be the reserve prices, even when the clock prices reached very high levels in the clock stage. The proposed ECCA design does not share this drawback. The next proposition calculates a lower bound on ECCA revenue after the primary stage.

Proposition 6. *Suppose that the auctioneer uses an exposure formula that is no more aggressive than the upper exposure formula. Then the auction revenue in the ECCA is bounded from below by:*

(a) If there exists at least one bidder $i \in N$ such that $d_i(T) \geq p(T)\,U$, then

$$R \;\geq\; p(T)\,S \;-\; \sum_{j \in N} d_j(T) \;+\; (N-1)\,gv_N(T,S),$$

(b) If $d_i(T) < p(T)\,U$ for all bidders $i \in N$, then

$$R \;\geq\; \min \left\{ \; p(T)\, x_N(T) \;\; , \quad p(T)\, S - \sum_{j \in N} d_j(T) + (N-1)\, gv_N(T,S) \; \right\}.$$

where $d_i(T)$ is the discount of bidder i from the exposure formula (5.1).

5.3 Alternative Exposure-Based Auction Designs

In this section we introduce two additional practical auction designs that are motivated by the theory developed in Section 3. These designs are variations on the CCA and SMRA, respectively, the most popular auction designs for modern spectrum auctions.

Each of these alternative designs utilizes the clock auction procedure developed in Section 3.1. In particular, the auctioneer uses a continuous price path $p(.)$ on $[0,T]$ to collect demand information from bidders. All demand choices of bidders are constrained by the GARP activity rule, and each bidder is informed about its current exposure level through the exposure discount. The clock procedure terminates at time T, reaching a feasible allocation $x(T) = (x_1(T), ..., x_N(T))$. Section 3.2 describes a general framework for conducting the secondary stage. The auctioneer can customize the rules of the secondary stage to obtain auction designs with somewhat different properties. For example, the auctioneer might introduce extra activity rules for bids received in the supplementary stage, impose extra constraints on the solution of the winner determination problem, or even impute bids on behalf of bidders for various bundles using the information collected in the primary stage and completely skip the "live" bidding.

5.3.1 ECCA with Winnings Protection

A common critique of the standard CCA design is the allocation uncertainty faced by the bidder in the supplementary round: the risk of winning nothing in the auction — i.e., the risk that the final winner determination selects the null package for a bidder, rather than one of his actual (clock or supplementary) bids. The supplementary round in the CCA has two main goals: (1)

increasing the value of the allocation; and (2) collecting bids that are used to calculate opportunity costs. Note that any attempt to reduce the allocation uncertainty in the CCA is in fundamental conflict with the second goal, since any reduction in the allocation uncertainty distorts bidder incentives for expressing their opportunity costs during the supplementary round.

In contrast, the sole goal of the secondary stage in the ECCA is to increase the value of the allocation. Therefore, no conflict arises if the auctioneer decides to reduce the uncertainty. In the ECCA design specified above, a bidder can completely avoid the losing outcome if she is willing to place a bid on the final clock package at the upper exposure level or above. However, the upper exposure amount can be a relatively high multiple of the final clock bid, especially for a small bidder. To provide greater certainty about the auction outcome, the auctioneer can adopt additional measures that place greater restrictions on the eventual outcome, for example by adopting a lower bid amount that precludes the losing outcome ("protection"). Such an intervention can potentially lead to an inefficient outcome. However, if the auctioneer believes that the values come from a restricted domain that does not conflict with the protective measures, the protective measures may be well justified.

> **ECCA with Winnings Protection:** The primary stage follows Section 3.1. If needed, the secondary stage follows Section 3.2 and, in addition, a protection amount W_i is specified for each bidder i for whom $x_i(T) \neq 0$. The winning allocation $x^* = (x_1^*, ..., x_n^*)$ is determined by maximizing $\sum_N b_j(x_j)$ such that: (1) $x \in X$; and (2) $x_i^* \neq 0$ for bidder i if $b_i(x_i(T)) \geq W_i$. Bidder payments $y^* = (y_1^*, ..., y_n^*)$ are determined by (3.23).

5.3.2 SMRA with Vickrey Pricing

Instead of having a secondary stage with "live" bidding, the auctioneer can completely automate the secondary stage and administratively assign all unallocated lots to bidders. Suppose that all units are demanded by bidders at the beginning of the primary stage. During the clock rounds, the auctioneer can tentatively assign units of the product with excess supply to bidders who are "responsible" for creating excess supply. This can be done by processing clock bids received at time t one at a time in some order. Then, each bidder is informed about lots $u_i(t) \geq 0$ that will ultimately be assigned to bidder i at the end of the auction if nobody claims them before that. The secondary stage is fully automated and occurs instantaneously. For each bidder i, the auctioneer adds one bid for bundle $z_i(T) = x_i(T) + u_i(T)$ which represents the clock demand of bidder i at time T plus all lots that bidder i is responsible for at time T. The bid for bundle $z_i(T)$ is placed at the maximum possible level consistent with GARP, i.e.,

$$b_i(z_i(T)) \ = \ p(T)\, z_i(T) \ - \ gv_i(T, z_i(T)) \tag{5.4}$$

This auction design has some basic similarities to the standard SMRA format. In particular, the auction stops being a package auction since bidders can end up winning undesired lots. But, unlike the SMRA, which traditionally operates as an auction with uniform prices, this design uses opportunity-cost-based pricing.

One way to conceptualize the administrative assignment is to view it as a built-in "bid withdrawal" feature. Bid withdrawals are a common element of the SMRA design that partially addresses the bidders' aggregation risk. Bid withdrawals typically work as follows: a bidder is permitted to withdraw some of its provisionally-winning bids, at the risk of being subject to a financial penalty, and to use the associated eligibility to bid for other lots. The penalty is based upon the difference between the price at which the withdrawal was made and the ultimate selling price of these lots. In this sense, the administrative assignment works similarly to bid withdrawals, with two key differences. First, with withdrawals, the bidder can end up paying for lots that she does not receive. With the administrative assignment, the bidder is allocated $u_i(T)$ in addition to $x_i(T)$. Second, with withdrawals, the penalty can be as high as the price at which the bid was withdrawn, while with the administrative assignment, the price that the bidder pays for extra lots is limited to the maximum possible value consistent with the bidder's bidding history. Therefore, the administrative assignment can be viewed as a milder version of the bid withdrawals that are commonplace in SMRAs.

> **SMRA with Vickrey Pricing:** The primary stage follows Section 3.1, and in addition, all products with excess supply are tentatively assigned to bidders using some process. The winning allocation of bidder i is given by $x_i^* = z_i(T)$. Bidder payments $y^* = (y_1^*, ..., y_n^*)$ are determined by (3.23) where it is assumed that the bid amount for package x_i^* implicitly made by bidder i is $b_i(x_i^*) = p(T)\, x_i^* - gv_i(T, x_i^*)$.

6 Conclusion

Dynamic implementations of the VCG mechanism in various settings have received a great deal of attention in the literature. For the general setting with heterogeneous goods, quasilinear utilities and private values, the Vickrey outcome cannot be attained in general if the auctioneer is limited to using linear and anonymous prices to elicit bidders' preferences. However, most practical auction applications rely on simple price trajectories.

In this paper, we have developed a way to approximate the Vickrey outcome when bidding data is generated by a clock auction that uses a linear and anonymous price trajectory. We establish efficiency bounds on the attained allocation and we introduce several exposure calculations that approximate the VCG payments. The exposure calculations utilize the Generalized Axiom of Revealed Preference (GARP).

The resulting auction design, the ECCA, resolves many of the issues associated with existing formats for spectrum auctions. In particular, it is a package auction that yields approximately opportunity-cost pricing, thereby eliminating the major strategic issues present in the SMRA. At the same time, it operates as an iterative first-price auction, ameliorating the price uncertainty and the known strategic issues of the CCA.

References

[1] Afriat, S.N. (1967), "The Construction of a Utility Function from Expenditure Data," *International Economic Review*, 8(1), pp. 67-77.

[2] Ausubel, Lawrence M. (2004), "An Efficient Ascending-Bid Auction for Multiple Objects," *American Economic Review*, 94(5), pp. 1452-1475.

[3] Ausubel, Lawrence M. (2006), "An Efficient Dynamic Auction for Heterogeneous Commodities," *American Economic Review*, 96(3), pp. 602-629.

[4] Ausubel, Lawrence M., and Oleg Baranov (2014), "Market Design and Evolution of the Combinatorial Clock Auction," *American Economic Review: P&P*, 104(5), pp. 446-451.

[5] Ausubel, Lawrence M., Peter Cramton and Paul Milgrom (2006). "The clock-proxy auction: a practical combinatorial auction design," *in (P. Cramton, Y. Shoham and R. Steinberg, eds.), Combinatorial Auctions*, pp. 115-138, Cambridge: MIT Press.

[6] Ausubel, Lawrence M., and Paul Milgrom (2002), "Ascending Auctions with Package Bidding," *Frontiers of Theoretical Economics*, 1(1), pp. 1-42.

[7] Baranov, Oleg, Lawrence M. Ausubel, Christina Aperjis and Thayer Morrill (2017). "Efficient Procurement Auctions with Increasing Returns." *American Economic Journal: Microeconomics*, 9(3), pp. 1-27

[8] Bichler, Martin, Pasha Shabalin and Jürgen Wolf (2013). "Do core-selecting combinatorial clock auctions always lead to high efficiency? An experimental analysis of spectrum auction designs." *Experimental Economics*, 16(4), pp. 511-545

[9] Bichler, Martin, Jacob Goeree, Stefan Mayer and Pasha Shabalin (2014). "Spectrum auction design: Simple auctions for complex sales." *Telecommunications Policy*, 38(7), pp. 613-622

[10] Bichler, Martin, and Jacob Goeree (2017), "Frontiers in spectrum auction design," *International Journal of Industrial Organization*, 50, pp. 372-391.

[11] Binmore, Ken, and Paul Klemperer (2002), "The Biggest Auction Ever: The Sale of the British 3G Telecom Licences," *The Economic Journal*, 112, pp. 74-96.

[12] Bikhchandani, Sushil, and Joseph M. Ostroy (2002), "The package assignment model," *Journal of Economic Theory*, 107(2), pp. 377-406.

[13] Bikhchandani, Sushil, and Joseph M. Ostroy (2006), "Ascending price Vickrey auctions," *Games and Economic Behavior*, 55(2), pp. 215-241.

[14] Bulow, Jeremy, Jonathan Levin, and Paul Milgrom (2017), "Winning Play in Spectrum Auctions." *Handbook of Spectrum Auction Design*

[15] Clarke, Edward H (1971). "Multipart Pricing of Public Goods." *Public Choice*, 11, pp. 17-33.

[16] Cramton, Peter (1995), "Money Out of Thin Air: The Natiowide Narrowband PCS Auction", *Journal of Economics & Management Strategy* , 4, pp. 267-343.

[17] Cramton, Peter (2013), "Spectrum Auction Design", *Review of Industrial Organization*, 42(2), pp. 161-190.

[18] Demange, Gabrielle, David Gale and Marilda Sotomayor (1986), "Multi-Item Auctions," *Journal of Political Economy*, 94(4), pp. 863-872.

[19] de Vries, Sven, James Schummer and Rakesh V. Vohra (2007), "On ascending Vickrey auctions for heterogeneous objects," *Journal of Economic Theory*, 132(1), pp. 95-118.

[20] Gretschko, Vitali, Stephan Knapek and Achim Wambach (2017), "Bidding Complexities in Combinatorial Clock Auctions." *Handbook of Spectrum Auction Design, Cambridge University Press*

[21] Groves, Theodore (1973), "Incentives in Teams," *Econometrica*, 41(4), pp. 617-31.

[22] Gul, Faruk, and Ennio Stacchetti (2000), "The English Auction with Differentiated Commodities," *Journal of Economic Theory*, 92(1), pp. 66-95.

[23] Janssen, Maarten, and Vladimir Karamychev (2016), "Spiteful bidding and gaming in combinatorial clock auctions," *Games and Economic Behavior*, 100(1), pp. 186-207.

[24] Janssen, Maarten, and Bernhard Kasberger (2017), "On the Clock of the Combinatorial Clock Auction," *working paper*.

[25] Lamy, Laurent (2012), "On minimal ascending auctions with payment discounts," *Games and Economic Behavior*, 75, pp. 990-999.

[26] Levin, Jonathan, and Andrzej Skrzypacz (2016), "Properties of the Combinatorial Clock Auction," *American Economic Review*, 106(9), pp. 2528-2551.

[27] McAfee, R. Preston, and John McMillan (1996), "Analyzing the Airwaves Auction," *Journal of Economic Perspectives*, 10(1), pp. 159-175.

[28] McMillan, John (1994), "Selling Spectrum Rights," *Journal of Economic Perspectives*, 8(3), pp. 145-162.

[29] Milgrom, Paul (2000), "Putting Auction Theory to Work: The Simultaneous Ascending Auction," *Journal of Political Economy*, 108(2), pp. 245-272.

[30] Milgrom, Paul (2004), "Putting Auction Theory to Work," *Cambridge: Cambridge University Press.*

[31] Mishra, Debasis, and David C. Parkes (2007), "Ascending Price Vickrey Auctions for General Valuations," *Journal of Economic Theory*, 132, pp. 335-366.

[32] Parkes, David C., and Lyle H. Ungar (2000), "Interative Combinational Auctions: Theory and Practice," In *Proceedings of the 17th National Conference on Artificial Intelligence* , (AAAI 2000), pp. 74-81.

[33] Parkes, David C., and Lyle H. Ungar (2002), "An Ascending-Price Generalized Vickrey Auction," *Discussion paper, Harvard University*

[34] Parkes, David C. (2006), "Interative Combinational Auctions," *In Combinatorial Auctions, eds. Peter Cramton, Yoav Shoham, and Richard Steinberg. Cambridge: MIT Press.*, pp. 41-78

[35] Rochet, Jean-Charles (1987), "A Necessary and Sufficient Condition for Rationalizability in a Quasi-Linear Context," *Journal of Mathematical Economics*, 16, pp. 191-200.

[36] Vickrey, William (1961), "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, 16(1), pp. 8-37.

For Online Publication

A Appendix - Aggregate Exposure

This appendix contains all results related to the upper and lower exposure formulas that are calculated using only the aggregate demand information.

Define an alternative notion of the GARP violation for coalition of bidders that is based on the aggregate demand. For coalition M, *the joint GARP violation based on the aggregate demand* when jointly bidding for bundle Z at time t is given by:

$$gv_M^A(t,Z) = \max_{s\in[0,t]} \left\{ p(s)[x_M(s) - Z] + \int_s^t p(u)dx_M(u) + p(t)[Z - x_M(t)] \right\} \quad \text{(A.1)}$$

Naturally, the amount of information contained in individual demands is larger than the the amount of information captured by the aggregate demand – it is possible to violate individual GARPs without violating the aggregate GARP. Hence, the joint GARP violation would be at least as small for the aggregated version than for the version based on individual demands.

Lemma 5. *Suppose that $x_i(t)$ satisfies GARP on $[0,T]$ for all $i \in M$. Then*

$$gv_M^A(t,Z) \leq gv_M(t,Z) \qquad \forall Z \in \Omega \qquad \forall t \in [0,T] \quad \text{(A.2)}$$

Now we state the direct analogs of Theorems 2 and 3 that calculate the upper and lower exposure using the aggregate demand as input.

Theorem 2′. *If all bidders in N_{-i} bid truthfully in the primary stage, then*

$$y_i^Q(x(T)) \ \leq \ UE_i(T) \ \leq \ UE_i^A(T)$$

where $UE_i^A(T)$ is the upper exposure of bidder i based on the aggregate demand given by

$$UE_i^A(T) \ = \ \min_{t\in[0,T]} \left\{ p(t)[S - x_{-i}(t)] - \int_t^T p(u)dx_{-i}(u) \right\}.^9 \quad \text{(A.3)}$$

[9] Note that formula (A.3) is similar to the crediting/debiting formula from Ausubel

The upper exposure of bidder i based on the aggregate demand can be alternatively stated as:

$$UE_i(T) \; = \; p(T)\,[x_i(T) + U] \; - \; gv^A_{-i}(T, S) \tag{A.4}$$

For the lower exposure, first define an aggregate version of T_M as:

$$T^A_M = \big\{t \in [0, T] : x_M(t) \leq S\big\} \tag{A.5}$$

Theorem 3′. *If all bidders in N_{-i} bid truthfully in the primary stage, then*

$$LE^A_i(T) \; \leq \; LE_i(T) \; \leq \; y^Q_i(x_i(T)) \tag{A.6}$$

where $LE^A_i(T)$ is the lower exposure of bidder i based on aggregate demand given by

$$LE^A_i(T) \; = \; \max_{t \in T^A_{-i}} \left[- \int_t^T p(u) dx_{-i}(u) \right]. \tag{A.7}$$

The lower exposure of bidder i based on aggregate demand can be alternatively stated as:

$$LE^A_i(T) \; = \; p(T)\,[x_i(T) + U] \; - \; \big(gv^A_{-i}(T, x_{-i}(\hat{t})) \; + \; p(T)[S - x_{-i}(\hat{t})]\big) \tag{A.8}$$

where $\hat{t}$ solves (A.7) for bidder i.

To state the analogs of Theorem 4 and Corollary 1, fist define an aggregate version of Ψ_M as:

$$\Psi^A_M \; = \min_{t \in T^A_M} \; p(t)\,[S - x_M(t)]. \tag{A.9}$$

Note that $\Psi_M \leq \Psi^A_M$ for any coalition $M \subseteq N$ by Lemma 3.

Theorem 4′. *If all bidders in N_{-i} bid truthfully in the primary stage, then:*

$$UE^A_i(T) \; - \; LE^A_i(T) \; \leq \; \Psi^A_{-i}. \tag{A.10}$$

(2006):

$$a_i(T) = p(0)\,[S - x_{-i}(0)] - \int_0^T p(u) dx_{-i}(u).$$

Corollary 1′.

(a) Suppose that all bidders in N_{-i} bid truthfully in the primary stage. If there exists $t \in [0,T]$ such that $x_{-i}(t) = S$, then

$$LE_i^A(T) \;=\; LE_i(T) \;=\; y_i^Q(x_i(T)) \;=\; UE_i(T) \;=\; UE_i^A(T).$$

(b) If there are only two bidders and both of them bid truthfully in the primary stage, then

$$LE_i^A(T) \;=\; LE_i(T) \;=\; y_i^Q(x_i(T)) \;=\; UE_i(T) \;=\; UE_i^A(T) \qquad i = 1,2$$

B Appendix - Proofs

PROOF OF PROPOSITION 1:

$$\begin{aligned} y_i^Q(x) &= w(N_{-i}) - \textstyle\sum_{N_{-i}} v_j(x_j^*) - \sum_{N_{-i}} v_j(x_j) + \sum_{N_{-i}} v_j(x_j^*) \\ &= y_i^V + \textstyle\sum_{N_{-i}} [v_j(x_j^*) - v_j(x_j)] = y_i^V + \Delta v_{-i}(x) \\ \pi_i^Q(x) &= v_i(x_i) - y_i^Q(x) = v_i(x_i) - v_i(x_i^*) + [v_i(x_i^*) - y_i^V] - \Delta v_{-i}(x) \\ &= \pi_i^V - \Delta v_N(x) \end{aligned}$$

□

PROOF OF AFRIAT'S THEOREM:.

If bidder i bids straightforwardly according to some $v(.)$, then by Lemma 2(a)

$$v(x_i(t)) = v(x_i(T)) - \int_t^T p(u)dx_i(u) \qquad \forall t \in [0,T] \tag{B.1}$$

and for any $t, t' \in [0,T]$:

$$\begin{aligned} & v(x_i(t)) - p(t)x_i(t) \geq v(x_i(t')) - p(t)x_i(t') \\ & p(t)[x_i(t) - x_i(t')] - \textstyle\int_{t'}^T p(u)dx_i(u) + \int_t^T p(u)dx_i(u) \leq 0 \qquad \text{(B.2)} \\ & p(t)[x_i(t) - x_i(t')] + \textstyle\int_t^{t'} p(u)dx_i(u) \leq 0 \end{aligned}$$

For the converse, construct function $v(.)$ using (B.1) above and setting $v(x_i(T) := p(t)\, x_i(T)$. For this $v(.)$, (B.2) applies in the opposite direction so $v(.)$ rationalizes $x_i(.)$ on $[0,T]$. □

PROOF OF LEMMA 1:

(a): Suppose that there exist $s' \in [0,t]$ such that $x_i(s') \neq z$ and the right-hand side of (2.6) is strictly higher for s' than for $s \in [0,t]$ such that $x_i(s) = z$. Then demands at s and s' violate GARP since:

$$\begin{aligned} p(s')[x_i(s') - z] + \int_{s'}^{t} p(u)dx_i(u) + p(t)[z - x_i(t)] &> \int_{s}^{t} p(u)dx_i(u) + p(t)[z - x_i(t)] \\ p(s')[x_i(s') - x_i(s)] + \int_{s'}^{s} p(u)dx_i(u) &> 0 \end{aligned}$$

(b): Let s and s' denote time when the right-hand side of (2.6) is maximized for packages z and z'. Then

$$(p(t) - p(s))[z' - z] \;\leq\; gv_i(z',t) - gv_i(z,t) \;\leq\; (p(t) - p(s'))[z' - z]$$

For $z' \geq z$ and nondecreasing $p(.)$, we have

$$0 \;\leq\; gv_i(z',t) - gv_i(z,t) \;\leq\; p(t)[z' - z]$$

□

PROOF OF LEMMA 2:

(a): Denote $t_1, ..., t_m$ all times in the interval $[t,T]$ when bidder i switched its demand (all t's such that $x_i(t' - 0) \neq x_i(t')$). Given the continuous price path $p(.)$, bidder i is indifferent at all switch points, i.e.,

$$\begin{aligned} v(x_i(t)) - p(t_1)\, x_i(t) &= v(x_i(t_1)) - p(t_1)\, x_i(t_1) \\ ... \quad & ... \quad ... \\ v(x_i(t_{m-1})) - p(t_m)\, x_i(t_{m-1}) &= v(x_i(T)) - p(t_m)\, x_i(T) \end{aligned}$$

Then

$$\begin{aligned} v(x_i(t)) &= v(x_i(T)) - p(t_1)[x_i(t_1) - x_i(t)] + ... + p(t_m)[x_i(T) - x_i(t_{m-1})] \\ &= v(x_i(T)) - \int_{t}^{T} p(u)dx_i(u) \end{aligned}$$

The equivalent formulation with $gv_i(T, x_i(t))$ is obtained by applying Lemma 1(a).

(b):

$$\begin{aligned} v(z) &\leq \min_{s\in[0,T]} \{v(x_i(s)) + p(s)[z - x_i(s)]\} \\ &= \min_{s\in[0,T]} \left\{v(x_i(T)) - \int_{s}^{T} p(u)dx_i(u) + p(s)[z - x_i(s)]\right\} \\ &= v(x_i(T)) + \min_{s\in[0,T]} \left\{p(s)[z - x_i(s)] - \int_{s}^{T} p(u)dx_i(u)\right\} \\ &= v(x_i(T)) + p(T)[z - x_i(T)] - gv_i(T,z) \end{aligned}$$

□

PROOF OF THEOREM 1:

Since the clock prices of goods with excess demand increase with the speed bounded away from zero, the aggregate demand of truthful bidders drops below supply in finite time. Denote $x^* = (x_1^*, ..., x_n^*)$ an efficient allocation. Then

$$\begin{aligned}
\Delta v_N(x(T)) &= \sum_{j \in N} [v_j(x_j^*) - v_j(x_j(T))] \\
&\leq \sum_{j \in N} \left[p(T)(x_j^* - x_j(T)) - gv_j(T, x_j^*)\right] \qquad \text{by Lemma 2}(b) \\
&= p(T)[S - x_N(T)] - \sum_{j \in N} gv_j(T, x_j^*) \\
&\leq p(T)\, U - gv_N(T, S)
\end{aligned}$$

If $U = 0$, then $gv_N(T, S) = 0$ and $\Delta v_N(x(T)) = 0$. Then allocation $x(T)$ is efficient and $y^Q(x(T)) = y^V$.

□

PROOF OF THEOREMS 2 AND 2′:

$$
\begin{aligned}
y_i^Q(x(T)) &= w(N_{-i}) - \sum_{j\neq i} v_j(x_j(T)) \\
&= \max_{\sum\limits_{j\neq i} z_j = S} \left\{ \sum_{j\neq i} [v_j(z_j) - v_j(x_j(T))] \right\} \\
&\leq \max_{\sum\limits_{j\neq i} z_j = S} \left\{ \sum_{j\neq i} \left[\min_{t_j\in[0,T]} \left\{ p(t_j)[z_j - x_j(t_j)] - \textstyle\int_{t_j}^T p(u)dx_j(u) \right\} \right] \right\} && = UE_i(T) \\
&\leq \max_{\sum\limits_{j\neq i} z_j = S} \left\{ \min_{t\in[0,T]} \left\{ \sum_{j\neq i} \left[p(t)[z_j - x_j(t)] - \textstyle\int_t^T p(u)dx_j(u) \right] \right\} \right\} \\
&= \min_{t\in[0,T]} \left\{ p(t)[S - x_{-i}(t)] - \textstyle\int_t^T p(u)dx_{-i}(u) \right\} && = UE_i^A(T)
\end{aligned}
$$

$$
\begin{aligned}
UE_i(T) &= \max_{\sum\limits_{j\neq i} z_j = S} \left\{ \sum_{j\neq i} \left[\min_{t_j\in[0,T]} \left\{ p(t_j)[z_j - x_j(t_j)] - \textstyle\int_{t_j}^T p(u)dx_j(u) \right\} \right] \right\} \\
&= -\min_{\sum\limits_{j\neq i} z_j = S} \left\{ \sum_{j\neq i} \left[\max_{t_j\in[0,T]} \left\{ p(t_j)[x_j(t_j) - z_j] + \textstyle\int_{t_j}^T p(u)dx_j(u) \right\} \right] \right\} \\
&= -\min_{\sum\limits_{j\neq i} z_j = S} \left\{ \sum_{j\neq i} [gv_j(T, z_j) - p(T)[z_j - x_j(T)]] \right\} \\
&= p(T)[S - x_{-i}(T)] - \min_{\sum\limits_{j\neq i} z_j = S} \left\{ \sum_{j\neq i} gv_j(T, z_j) \right\} \\
&= p(T)\,[x_i(T) + U] - gv_{-i}(T, S)
\end{aligned}
$$

$$
\begin{aligned}
UE_i^A(T) &= \min_{t\in[0,T]} \left\{ p(t)[S - x_{-i}(t)] - \textstyle\int_t^T p(u)dx_{-i}(u) \right\} \\
&= -\max_{t\in[0,T]} \left\{ p(t)[x_{-i}(t) - S] + \textstyle\int_t^T p(u)dx_{-i}(u) \right\} \\
&= p(T)[S - x_{-i}(T)] - gv_{-i}^A(T, S) \\
&= p(T)\,[x_i(T) + U] - gv_{-i}^A(T, S)
\end{aligned}
$$

□

PROOF OF THEOREMS 3 AND 3′:

$$
\begin{aligned}
y_i^Q(x(T)) &= w(N_{-i}) - \sum_{j \neq i} v_j(x_j(T)) \\
&\geq \max_{\{t_j\} \in T_{-i}} \left\{ \sum_{j \neq i} [v_j(x_j(t_j)) - v_j(x_j(T))] \right\} \\
&= \max_{\{t_j\} \in T_{-i}} \left\{ - \sum_{j \neq i} \int_{t_j}^{T} p(u) dx_j(u) \right\} & = LE_i(T) \\
&\geq \max_{t \in T_{-i}^A} \left\{ - \sum_{j \neq i} \int_{t}^{T} p(u) dx_j(u) \right\} \\
&= \max_{t \in T_{-i}^A} \left\{ - \int_{t}^{T} p(u) dx_{-i}(u) \right\} & = LE_i^A(T)
\end{aligned}
$$

$$
\begin{aligned}
LE_i(T) &= - \sum_{j \neq i} \int_{\hat{t}_j}^{T} p(u) dx_j(u) \\
&= \sum_{j \neq i} \left[p(T)[x_j(\hat{t}_j) - x_j(T)] - gv_j(T, x_j(\hat{t}_j)) \right] \\
&= p(T)[x_i(T) + U] - \left(\sum_{j \neq i} gv_j(T, x_j(\hat{t}_j)) + p(T) \left[S - \sum_{j \neq i} x_j(\hat{t}_j) \right] \right) \\
LE_i^A(T) &= - \int_{\hat{t}}^{T} p(u) dx_{-i}(u) \\
&= p(T)[x_{-i}(\hat{t}) - x_{-i}(T)] - gv_{-i}^A(T, x_{-i}(\hat{t})) \\
&= p(T)[x_i(T) + U] - (gv_{-i}^A(T, x_{-i}(\hat{t})) + p(T)[S - x_{-i}(\hat{t})])
\end{aligned}
$$

□

PROOF OF LEMMA 3:

$$
\begin{aligned}
\Psi_M &= \max_{\sum_{j \in M} z_j = S} \; \min_{\{t_j\} \in T_M} \left[\sum_{j \in M} p(t_j) \left[z_j - x_j(t_j) \right] \right] \\
&\leq \max_{\sum_{j \in M} z_j = S} \; \min_{t:\, x_M(t) \leq S} \left[\sum_{j \in M} p(t) \left[z_j - x_j(t) \right] \right] \\
&= \min_{t:\, x_M(t) \leq S} \left[p(t) \left[S - x_M(t) \right] \right]
\end{aligned}
$$

□

PROOF OF THEOREMS 4 AND 4′:

$$\begin{aligned}
UE_i(T) - LE_i(T) &= \max_{\sum_{j\neq i} z_j = S} \left\{ \sum_{j\neq i} \left[\min_{t_j \in [0,T]} \left\{ p(t_j)[z_j - x_j(t_j)] - \int_{t_j}^{T} p(u)dx_j(u) \right\} \right] \right\} + ... \\
&\quad ... + \min_{\{t_j\} \in T_{-i}} \left\{ \sum_{j\neq i} \int_{t_j}^{T} p(u)dx_j(u) \right\} \\
&\leq \max_{\sum_{j\neq i} z_j = S} \left\{ \min_{\{t_j\} \in T_{-i}} \left\{ \sum_{j\neq i} \left[p(t_j)[z_j - x_j(t_j)] - ... \right.\right.\right. \\
&\quad \left.\left.\left. ... - \int_{t_j}^{T} p(u)dx_j(u) + \int_{t_j}^{T} p(u)dx_j(u) \right] \right\} \right\} \\
&= \Psi_{-i}
\end{aligned}$$

$$\begin{aligned}
UE_i^A(T) - LE_i^A(T) &= \min_{t\in[0,T]} \left\{ p(t)[S - x_{-i}(t)] - \int_{t}^{T} p(u)dx_{-i}(u) \right\} + ... \\
&\quad ... + \min_{t \in T_{-i}^A} \left\{ \int_{t}^{T} p(u)dx_{-i}(u) \right\} \\
&\leq \min_{t \in T_{-i}^A} \left\{ p(t)[S - x_{-i}(t)] - \int_{t}^{T} p(u)dx_{-i}(u) + \int_{t}^{T} p(u)dx_{-i}(u) \right\} \\
&= \Psi_{-i}^A
\end{aligned}$$

□

PROOF OF COROLLARIES 1 AND 1′:

(a): If there exists $t' \in [0,T]$ such that $x_{-i}(t) = S$, then $\Psi_{-i} = \Psi_{-i}^A = 0$.

(b): If $n = 2$ and bidders are bidding truthfully, then $x_{-i}(0) = S$ for $i = 1, 2$ since $p(0) = 0$ and monotonicity assumption (A3). □

PROOF OF PROPOSITION 2:

Suppose that bid $b_i(z)$ for bundle $z = x_i$ is a potentially winning bid. Then:

$$\begin{aligned}
\sum_{j\in N} b_j(x_j) &\geq \sum_{j\in N} b_j(x_j(T)) \\
b_i(z) - b_i(x_i(T)) &\geq \sum_{j\neq i} [b_j(x_j(T)) - b_j(x_j)] \\
&\geq \sum_{j\neq i} \left[p(T)\,[x_j(T) - x_j] + gv_j(T, x_j) \right] \\
&\geq p(T)\,[z - x_i(T)] - p(T)\,U + gv_{-i}(T, S - z)
\end{aligned}$$

The last inequality is due to Lemma 1(b). For the converse, denote allocation $x = (x_1, ..., x_n)$ where $x_i = z$ be the feasible allocation such that $gv_{-i}(T, S - z) = \sum_{j \neq i} gv_j(T, x_j)$ and set $b_j(x_j) - b_j(x_j(T)) = p(T)\,[x_j - x_j(T)] - gv_j(T, x_j)$ for all bidders $j \in N_{-i}$.

□

PROOF OF THEOREM 5:

First, denote $x^* = (x_1^*, ..., x_n^*)$ the efficient allocation that maximizes $\sum_N v_j(x_j)$ on X. If all bidders $i \in N$ were to bid $b_i(x_i^*) = b_i(x_i(T)) + mv_i(x_i^*)$ (allowed by the activity rule), then

$$\sum_N [b_j(x_j^*) - b_j(x_j(T))] = \sum_N [v_j(x_j^*) - v_j(x_j(T))] \geq 0$$

Then $b_i(x_i^*)$ is a potentially winning bid for each bidder i, and, under semi-truthful bidding, $\sum_N b_j(x_j^*) \geq \sum_N b_j(x_j(T))$.

Second, denote $\bar{X}$ a set of allocations $(x_1, .., x_n) \in X$ such that $\sum_N b_j(x_j) \geq \sum_N b_j(x_j(T))$. Note that $x^* \in \bar{X}$ under semi-truthful bidding. For any allocation $(x_1, .., x_n) \in \bar{X}$, $b_i(x_i)$ is a potentially winning bid for each $i \in N$. If $x_i \neq 0$, then $b_i(x_i) - b_i(x_i(T)) = v_i(x_i) - v_i(x_i(T))$ due to semi-truthful bidding. If $x_i = 0$, then $b_i(x_i(T)) < UE_i(T)$ by (3.18) and the tie-breaking criterion, and $b_i(x_i) - b_i(x_i(T)) = v_i(x_i) - v_i(x_i(T))$ due to semi-truthful bidding since $b_i(x_i(T)) = v_i(x_i(T))$ and $v_i(x_i) = b_i(x_i) = 0$. Then, $x^* = \arg\max_{x \in X} \sum_N b_j(x_j)$ since:

$$\max_{x \in X} \sum_{j \in N} b_j(x_j) \;\Leftrightarrow\; \max_{x \in X} \sum_{j \in N} [b_j(x_j) - b_j(x_j(T))] \;\Leftrightarrow\; \max_{x \in \bar{X}} \sum_{j \in N} [v_j(x_j) - v_j(x_j(T))]$$

Finally, for the efficient allocation $x^* \in \bar{X}$, we have

$$\Delta v_{-i}(x(T)) = \sum_{N_{-i}} [v_j(x_j^*) - v_j(x_j(T))] = \sum_{N_{-i}} [b_j(x_j^*) - b_j(x_j(T))] = \Delta b_{-i}(x(T)).$$

and $y_i^V = y_i^Q(x(T)) - \Delta b_{-i}(x(T))$ by Proposition 1(a).

□

PROOF OF PROPOSITION 4:

The auctioneer solves the following optimization problem:

$$\max_{x \in X} \sum_{j \in N} b_j(x_j) \Leftrightarrow \max_{x \in X} \sum_{j \in N} \left[b_j(x_j) - b_j(x_j(T)) \right] \Leftrightarrow \max_{x \in X} \; \Delta b_i(x_i(T)) + \Delta b_{-i}(x(T))$$

Bidder i's payoff from allocation x:

$$\begin{aligned} \pi_i(x_i) &= v(x_i) - \tilde{y}_i \\ &= v(x_i) - v_i(x_i(T)) + \Delta b_{-i}(x(T)) + v_i(x_i(T)) - E_i(T) \\ &= \Delta v_i(x_i(T)) + \Delta b_{-i}(x(T)) + v_i(x_i(T)) - E_i(T) \end{aligned}$$

Note that the last two terms in the payoff are constants. Then bidder $i's$ payoff is maximized by the auctioneer when she bids $\Delta b_i(x_i(T)) = \Delta v_i(x_i(T))$ for all bundles $z \in \Omega$ such that $b_i(z)$ is a potentially winning bid. □

PROOF OF LEMMA 4:

(a):

$$\begin{aligned} \tilde{y}_i &= E_i(T) - \Delta b_{-i}(x(T)) \\ &\geq p(T)\left[x_i(T) + U\right] - gv_{-i}(T, S) - \Delta b_{-i}(x(T)) \\ &\geq p(T)\left[x_i(T) + U\right] - gv_{-i}(T, S) - p(T)\left[S - x_{-i}(T)\right] + gv_{-i}(T, S) \\ &= 0 \end{aligned}$$

(b): For the winning allocation $x^* = (x_1^*, ..., x_n^*)$,

$$\begin{aligned} \sum_{j \in N} b_j(x_j^*) &\geq \sum_{j \in N} b_j(x_j(T)) \\ b_i(x_i^*) &\geq b_i(x_i(T)) - \Delta b_{-i}(x(T)) \\ &= E_i(T) - \Delta b_{-i}(x(T)) + [b_i(x_i(T)) - E_i(T)] \\ &= \tilde{y}_i + [b_i(x_i(T)) - E_i(T)] \end{aligned}$$

If $E_i(T) \leq b_i(x_i(T))$, then $\tilde{y}_i \leq b_i(x_i^*)$. □

PROOF OF PROPOSITION 5:

If $E_i(T) \leq p(T)\, x_i(T)$, then $E_i(T) \leq b_i(x_i(T))$. Given that $E_i(T) \geq UE_i(T)$, $0 \leq \tilde{y}_i \leq b_i(x_i^*)$ and $y_i^* = \tilde{y}_i$. □

PROOF OF THEOREM 6:

Since each player have bid truthfully in the primary stage, semi-truthful bidding is feasible for each bidder. Denote $x^* = (x_1^*, ..., x_n^*)$ a winning allocation of the auction when all bidders with an exception of bidder i bid semi-truthfully. Then for bidder i:

$$\begin{aligned} \tilde{y}_i &= E_i(T) - \Delta b_{-i}(x(T)) \\ &\geq y_i^Q(x(T)) - \Delta v_{-i}(x(T)) \\ &= y_i^Q(x^*) \end{aligned}$$

By construction, quasi Vickrey payments are nonnegative, so $\tilde{y}_i = y_i^Q(x^*) \geq 0$.

If $b_i(x_i(T)) \geq E_i(T)$, then $\tilde{y}_i \leq b_i(x_i^*)$ by Lemma 4(b). If $b_i(x_i(T)) < E_i(T)$, then $p(T)\, x_i(T) < E_i(T) = y_i^Q(x(T))$. Denote allocation x such that $\sum_{j \neq i}[v_j(x_j) - v_j(x_j(T))] = y_i^Q(x(T)) = E_i(T)$. It is easy to show that $b_j(x_j) = b_j(x_j(T)) + mv_j(x_j)$ is a potentially winning bid for each bidder $j \neq i$ (just assume that all bidders in N_{-i} bid in this way and bidder i does not bid at all in the secondary stage). Then, given the semi-truthful bidding by all bidders $j \neq i$ and $x_i^* \neq 0$:

$$\begin{aligned} \sum_{j \in N} b_j(x_j^*) &\geq \sum_{j \neq i} b_j(x_j) \\ b_i(x_i^*) + \Delta b_{-i}(x(T)) &\geq \sum_{j \neq i} \big[b_j(x_j) - b_j(x_j(T))\big] \\ &= \sum_{j \neq i} \big[v_j(x_j) - v_j(x_j(T))\big] \\ &= y_i^Q(x(T)) \;=\; E_i(T) \end{aligned}$$

In case $x_i^* \neq 0$, $b_i(x_i^*) \geq \tilde{y}_i$. Therefore, when opponents of bidder i bid semi-truthfully, $y_i^* = \tilde{y}_i$ for any bid profile submitted by bidder i such that $x_i^* \neq 0$. Then bidding semi-truthfully is a best response by Proposition 4.[10] □

PROOF OF THEOREM 7:

When all bidders bid truthfully in the primary stage and semi-truthfully in the secondary stage and $E_i(T) = y_i^Q(x(T))$, x^* is efficient and $y^* = \tilde{y} = y^V$ by Theorems 5 and 6. In addition, by Theorem 6, bidder i does not have profitable deviations in the secondary stage only. During the primary stage, any deviation of bidder i can

[10] Any deviation that leads to $x_i^* = 0$ is not profitable for bidder i since she gets $\pi_i^V \geq 0$ in the equilibrium.

potentially lead to an alternative price path $p'(.)$, the termination time T' and the final clock allocation $x'(T')$, and an alternative secondary stage. For any alternative history, $E_i(T') \geq UE_i(T') \geq y_i^Q(x'(T'))$ due to aggressive exposure formula. Denote $x^* = (x_1^*, ..., x_n^*)$ a winning allocation of this auction when all bidders with an exception of bidder i bid truthfully in the primary stage and semi-truthfully in the secondary stage. In the proof for Theorem 6, we showed that given semi-truthful bidding for opponents of bidder i in the secondary stage, bidder i's payment is either determined by $\tilde{y}_i = E_i(T') - \Delta b_{-i}(x'(T'))$ or bidder i wins bundle x_i^* only if $b_i(x_i^*) \geq \ y_i^Q(x'(T')) - \Delta b_{-i}(x'(T'))$. In either case,

$$\begin{aligned} y_i^* &= \min\{\tilde{y}_i, b_i(x_i^*)\} \\ &\geq y_i^Q(x'(T')) - \Delta v_{-i}(x'(T')) \\ &= y_i^Q(x^*) \end{aligned}$$

But then profit of bidder i after any deviation is $\pi_i^* \leq \pi_i^Q(x^*) \leq \pi_i^V$.

□

PROOF OF PROPOSITION 6:

The auction revenue is given by:

$$\begin{aligned} R &= \textstyle\sum_N y_j^* \\ &\geq \textstyle\sum_N \min\{\ E_j(T) - \Delta b_{-j}(x(T)),\ b_j(x_j^*)\ \} \\ &= \textstyle\sum_N [p(T)\, x_j(T) + \min\{\ p(T)\, U - d_j(T) - \Delta b_{-j}(x(T)), c_j + \Delta b_j(x(T))\ \}] \\ &= \textstyle\sum_N [p(T)\, x_j(T) - \Delta b_{-j}(x(T)) + \min\{\ p(T)\, U - d_j(T), c_j + \Delta b_N(x(T))\ \}] \\ &= p(T)\, x_N(T) - (N-1)\, \Delta b_N(x(T)) + \textstyle\sum_N \min\{p(T)\, U - d_j(T), c_j + \Delta b_N(x(T))\} \end{aligned}$$

To minimize auction revenue, set $c_j = 0$ for all $j \in N$. Note that the lower bound decreases with $\Delta b_N(x(T))$ as long as there is at least one bidder such that $p(T)U \leq d_j(T)$. In this case, the minimum is achieved when $\Delta b_N(x(T))$ is maximal (i.e., when $\Delta b_N(x(T)) = p(T)\, U - gv_N(T, S)$)

$$\begin{aligned} R &= p(T)\, x_N(T)\ -\ (N-1)\, \Delta b_N(x(T))\ + ... \\ &\quad + \sum_{j \in N} \min\big\{\ p(T)\, U - d_j(T),\ \Delta b_N(x(T))\ \big\} \\ &\geq p(T)\, x_N(T) + p(T)\, U + (N-1)\, gv_N(T, S) - \textstyle\sum_N d_j(T) \\ &= p(T)\, S - \sum_{j \in N} d_j(T) + (N-1)\, gv_N(T, S) \end{aligned}$$

If $p(T)U > d_j(T)$ for all bidders $j \in N$, then the auction revenue is first increasing and then decreasing with $\Delta b_N(x(T))$, and

$$R \;\geq\; \min \left\{ \; p(T)\, x_N(T),\; p(T)\, S - \sum_{j \in N} d_j(T) + (N-1)\, gv_N(T,S) \; \right\}.$$

$\square$

PROOF OF LEMMA 5:

$$\begin{aligned}
gv_M(t,Z) \;&= \min_{\sum_M z_j = Z} \left[\sum_{j \in M} gv_i(t, z_j) \right] \\
&= \min_{\sum_M z_j = Z} \left[\sum_{j \in M} \left[\max_{s \in [0,t]} \left\{ p(s)[x_j(s) - z_j] + \int_s^t p(u) dx_j(u) + p(t)[z_j - x_j(t)] \right\} \right] \right] \\
&\geq \min_{\sum_M z_j = Z} \left[\max_{s \in [0,t]} \left\{ \sum_{j \in M} \left[p(s)[x_j(s) - z_j] + \int_s^t p(u) dx_j(u) + p(t)[z_j - x_j(t)] \right] \right\} \right] \\
&= \min_{\sum_M z_j = Z} \left[\max_{s \in [0,t]} \left\{ p(s)[x_M(s) - Z] + \int_s^t p(u) dx_M(u) + p(t)[Z - x_M(t)] \right\} \right] \\
&= gv_M^A(t,Z)
\end{aligned}$$

$\square$