VCG, the Core, and Assignment Stages in Auctions

Lawrence M. Ausubel[*] and Oleg Baranov[†]

Current Draft: 25 June 2019

Preliminary and Incomplete

Abstract

The Vickrey-Clarke-Groves (VCG) mechanism is one of the most compelling constructs in mechanism design, but the presence of complementarities creates the theoretical possibility of non-core outcomes. In this paper, we empirically document such outcomes—including three zero-revenue rounds—in the Federal Communications Commission (FCC) Incentive Auction. We then model the two-stage approach commonly used for auctioning spectrum, where bidders bid first for generic blocks and then bid for specific frequencies. We demonstrate that this approach may induce complementarities in the second stage even when the underlying preferences are well-behaved. Unexpectedly, assignment stages prove to be likely environments for non-core outcomes.

Keywords: VCG mechanism, core, spectrum auction, assignment stage

[*] Department of Economics, University of Maryland, College Park, MD 20742, USA. Email: ausubel@econ.umd.edu

[†] Department of Economics, University of Colorado, Boulder, CO 80309, USA. Email: oleg.baranov@colorado.edu

1. Introduction

One of the most elegant and compelling constructs in the toolbox of mechanism design is the Vickrey-Clarke-Groves (VCG) mechanism.[1] In appropriate economic environments, asking agents to report their valuations for goods, allocating the goods efficiently relative to these reports, and compensating agents according to an opportunity cost calculation relative to these reports gives rise to an elegant mechanism. Truth-telling becomes a dominant strategy and efficiency is obtained.

However, almost two decades ago, economists added a cautionary amendment to our understanding.[2] Complementarities of goods may effectively create complementarities of agents, causing VCG payments to be uncompetitively low and to lie outside the core. They may also make the mechanism vulnerable to a host of other anomalies. Two winning bidders may reduce their payments by colluding to increase each of their bids. Two losing bidders may convert themselves into winning bidders and thereby increase their payoffs by coordinating on increasing their respective bids. A single agent can increase its payoff in the mechanism by placing some of its bids through a "shill" (i.e., a fictitious party).

Zero-revenue outcomes, the most extreme ramification of this problem, become a theoretical possibility. Consider the "local-local-global" model, with two goods (A and B) and three bidders (1, 2 and 3) possessing the following valuations:[3]

$v_1(A, B) = 2$; $v_1(A) = 0$; $v_1(B) = 0$; $v_1(\varnothing) = 0$;
$v_2(A, B) = 2$; $v_2(A) = 2$; $v_2(B) = 0$; $v_2(\varnothing) = 0$; and
$v_3(A, B) = 2$; $v_3(A) = 0$; $v_3(B) = 2$; $v_3(\varnothing) = 0$.

The global bidder 1 values goods A and B as perfect complements and values only the AB combination at 2. Local bidder 1 values only good A, at 2; and local bidder 2 values only good B, also at 2. In this classic example, good A is awarded to bidder 2 and good B is awarded to bidder 3, since this is the value-maximizing allocation. However, the price of each of the local bidders in the VCG mechanism is zero, since the social surplus with both local bidders present equals 4 while the social surplus with either local bidder absent equals 2, implying that each local bidder should keep the entire social surplus that it creates. Alternatively, the opportunity cost of allocating each local bidder its preferred good equals zero. As such, the VCG mechanism's revenues of zero are disappointing and the outcome is outside the core, since there is a blocking coalition (the seller and the global bidder) whose coalitional value equals 2.

Despite the theoretical literature, we are unaware of any documented empirical examples of zero-revenue outcomes. One possible reason for the dearth of known examples is that these observations spawned a literature on core-selecting mechanisms,[4] and real-world auctioneers, aware of this problem, typically include core adjustments in their rules. An alternative explanation for the dearth of examples is that zero-revenue outcomes are merely a theoretical curiosity and not a problem in practice.

In this paper, we document the occurrence of zero-revenue outcomes and non-core outcomes in what may initially appear to be an unlikely place: the assignment stage of a spectrum auction. Auctions of

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1 Vickrey (1961), Clarke (1971), Groves (1973).
2 Ausubel and Milgrom (2002, 2006).
3 Ausubel and Milgrom (2002) at pp. 5.
4 Day and Raghavan (2007), Day and Milgrom (2008), Day and Cramton (2012).

spectrum licenses that are close substitutes are often dichotomized into two phases. First, in the allocation stage, bidders bid for "generic" licenses (licenses that not attached to any specific frequency). Second, in the assignment stage, bidders express their relative preferences for different assignments of specific frequencies corresponding to their generic winnings from the allocation stage and the assignment that maximizes the sum of bidders' values, as expressed by the bids, is selected.

It is initially easy to overlook that complementarities would even be present in assignment stages. The presence of complementarities in mobile telephony is typically driven by scale and contiguity considerations. For example, the LTE technology, an important component of 4G service, is able to achieve substantially greater data transfer speeds using a single block of contiguous spectrum than using an equivalent bandwidth of non-contiguous spectrum. Thus, in an auction of four 10 MHz spectrum blocks that are identified (in ascending order of frequencies) as the A, B, C and D blocks and are otherwise equivalent, a winner of two blocks in the allocation stage would wish to be assigned the "AB" combination, the "BC" combination, or the "CD" combination. Being fully aware of this issue, an enlightened regulator would likely limit consideration to assignments in which winners of multiple blocks are assigned contiguous segments. As a result, the assignment stage lacks all typical sources of complementarities that are either contained in the allocation stage or eliminated with contiguity constraints.

We begin this paper by providing, in Section 2, some motivation for the two-stage approach of an allocation stage for generic spectrum followed by an assignment stage that assigns physical frequencies. Then, in Section 3, we document the occurrence of zero-revenue outcomes in the assignment stage of a very conspicuous, recent spectrum auction: the Federal Communication Commission (FCC) Broadcast Incentive Auction of 2016–17. In this auction, three rounds of the assignment stage produced zero-revenue outcomes and fully one-sixth of the assignment rounds yielded VCG outcomes that were outside the core.[5]

Then, in the rest of the paper, we attempt to explain why, contrary to initial expectations, an assignment stage is far from an unlikely place to find non-core outcomes.[6] In Section 4, we develop a stylized model of a two-stage Vickrey auction and introduce an equilibrium concept inspired by the dominant-strategy property of the VCG mechanism. Surprisingly—at least to us—we demonstrate that the decomposition of spectrum auctions into two stages can create bidder complementarities on its own. In particular, in Section 5, we show that the assignment stage can produce a non-core VCG outcome in settings that lack *any complementarities among goods* in the usual sense (e.g., where bidders' preferences exhibit gross substitutes).[7] Finally, we draw some implications for the design of future spectrum auctions in Section 6.

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[5] If one accounts for the fact that the configurations of winnings in some of the rounds made non-core VCG outcomes impossible (for example, if there were only two winners), the reported fraction of rounds in which the VCG outcome was outside the core rises from 16.7% to 18.4%.

[6] Indeed, the exercise of looking for non-core outcomes in an assignment stage may initially remind the reader of the allegory about the drunk who searches for his lost money under a street light, rather than in the dark alley where he lost it, because the lighting is better. Are we performing this exercise for the Incentive Auction only because it is "under the street light," i.e., because we have the data (and because it has received the most attention of any spectrum auction in the past decade)? The role of the second part of the paper is to show that the assignment stage of a spectrum auction is actually one of the more likely places to find non-core VCG outcomes.

[7] Kelso and Crawford (1982).

2. Motivating a two-stage approach

There are several reasons to take a two-stage approach to auctioning multiple closely-substitutable licenses. To aid exposition, consider an auctioneer who: (1) wants to efficiently allocate 20 spectrum licenses that are close substitutes; (2) does not enforce contiguity of winnings; and (3) uses a sealed-bid auction format such as the VCG mechanism.

With a one-stage approach, each bidder is required to bid on as many as $2^{20} \approx 1.05$ million different license combinations. This is impractical since bidders have to evaluate each bundle and communicate their values to the auctioneer. Moreover, even if bidders managed to bid on such a large number of packages, it would be infeasible for the auctioneer to solve for the efficient allocation. The winner determination problem (finding a feasible allocation that maximizes the sum of values) is NP-hard, and it is unsolvable in practice unless the number of bids is limited.

A popular practical solution to this computational problem is to restrict the number of bids that each bidder is allowed to submit to about 500 to 1000. Observe that such a restriction generally has efficiency implications, as each bidder needs to "guess" which 500 bids (out of 1 million) are the most relevant. Bidders who fail to coordinate on an efficient assignment might also fail to coordinate on an efficient allocation of license quantities (the fitting problem), producing a significant welfare loss.

In contrast, the two-stage approach elegantly sidesteps this computational problem. In the allocation stage, each bidder needs to bid for at most 20 different quantities. Provided that all bidders view these licenses as close substitutes, the efficient allocation of license quantities is obtained. In the assignment stage, bidders might still face a large number of bidding options (although significantly smaller relative to the one-stage approach), but the ramifications of bounding the number of bids are less critical here since the failure to coordinate on an assignment could never trigger an inefficient allocation of quantities.

Both bidding complexity and computational demands are dramatically reduced when winnings are restricted to contiguous assignments, which is also prevalent in practice. For example, with the same 20 licenses, the maximum number of combinations that a bidder has to consider falls from 1.05 million to just 210. At the same time, spectrum auctions routinely allocate licenses in different frequency bands and/or different geographic regions. If 20 licenses are divided into two bands of 10 licenses each, the number of combinations with contiguous assignment within each band grows to 3025, making the computational issues relevant again. Furthermore, if the two-stage approach is taken in a setting other than spectrum, there may be no analog to contiguity restrictions.

The two-stage approach further facilitates implementing a dynamic auction process. For example, the popular simultaneous multiple round auction (SMRA) format, which is dynamic but just a single stage, can lead to inordinately long auctions.[8] For example, with 20 licenses, it might take up to 20 rounds to increase the standing price on each license by just one price increment. More fundamentally, the SMRA does not support contiguity restrictions, enabling anti-competitive strategies such as when in one auction, "one bidder was attempting to assemble five adjoining licenses and the spoiler continued to bid

[8] The 2008 Canadian AWS auction extended 331 rounds and took 2 months. The 2014–15 US AWS-3 auction extended 441 rounds and took 2 ½ months.

on the middle license in a strategy that became known as 'giving the middle finger'."[9] Allowing bidders to bid quantities of generic spectrum produces shorter auctions, and imposing contiguity requirements in the assignment stage curtails anti-competitive strategies.

Perhaps the strongest argument against the one-stage approach in settings with near substitutes is the unnecessary complexity of bidding decisions faced by bidders. It is challenging to formulate a good model capturing bidding complexity for the same reasons as it is to develop a good model of bounded rationality. That is why we pin our argument on the computational limitations of the one-stage approach, which are well-understood in theory and practice. In any case, the US FCC and the Australian Communications and Media Authority (ACMA), both long-time users of the one-stage approach, have adopted two-stage procedures for many of their recent auctions.

3. The Assignment Stage in the FCC's Broadcast Incentive Auction

The Broadcast Incentive Auction included two separate but interdependent auctions: a reverse auction, where television broadcasters bid to relinquish their spectrum usage rights, and a forward auction, where wireless operators bid to acquire freed-up spectrum.[10] For the forward auction, the FCC adopted a two-stage approach. In the first stage, the FCC used a clock auction to award seven generic blocks of spectrum to bidders in 416 distinct partial economic areas (PEAS). In the second stage, bidders bid for physical frequency assignments in each of these regions.

The assignment stage was organized as a sequence of sealed-bid auctions, in six parallel sessions, where bidders bid for their assignments independently in each PEA, in descending order of population. To accelerate the process and to enhance contiguity, PEAs with the same winners and winnings were consolidated. As a result, the total number of assignment rounds was reduced from 416 to 228. Participating bidders were informed about their own assignments and payments in each region before bidding for their assignments in the next region.

FCC adopted the VCG mechanism for the assignment stage. Each winner of generic blocks was guaranteed a contiguous assignment within the region. Bidders were invited to bid on all possible contiguous assignments corresponding to their generic winnings, even though some of these assignments were incompatible with contiguity of other winners. This was done intentionally to avoid leaking information about generic winnings of participants.

The bidding data from the assignment stage of the Incentive Auction presents us with a unique opportunity to evaluate the extent of the core problem of the VCG mechanism. Treating bids as true values, we find that 38 instances (out of 228) where VCG payments were too low to be in the core. The cumulative revenue shortage across all assignment rounds is $4,411,699. That is, assuming the same set of bids, if the FCC had used a core-selecting payment rule, the assignment stage gross revenues would have been $140,342,331 instead of $135,930,632 (about 3.25% higher). However, given that core-selecting auctions are not incentive-compatible in general (i.e., bidders can gain by bidding in a non-truthful manner), the implications of this comparison are limited.

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[9] McAfee, McMillan, and Wilkie (2010), p. 181.

[10] A detailed description of the Incentive auction can be found in Aperjis, Ausubel and Baranov (2019). All publicly-available bidding data can be found at https://auctiondata.fcc.gov/public/projects/1000.

Most notably, there were three instances (out of the 38 instances with core violations) where bidders literally paid zero! To the best of our knowledge, this was the first time that the “zero revenue” scenario of the VCG mechanism was documented in the field. Next we provide details for one of these instances.

The assignment of seven blocks in PEAs 224 and 287 (De Kalb, IL and Kenosha, WI) was made in assignment round 37. In this consolidated region, Dish Network won one generic block, while T-Mobile and U.S. Cellular won three generic blocks each. All bids, including discarded incompatible bids, submitted by bidders in this round are reported in Table 1, with winning bids displayed in bold.

Table 1: Assignment Bids for PEAs 224 and 287 in $ million

Band Plan

A	B	C	D	E	F	G

Bidders (generic winnings)	Dish Network (1 block)	T-Mobile (3 blocks)	U.S. Cellular (3 blocks)
Compatible Bids (block(s), bid)	**A, 0**	ABC, 0.4m	ABC, 0
	D, 0	**BCD, 3.5m**	BCD, 0
	G, 0.237m	DEF, 0.010m	DEF, 4m
		EFG, 0	**EFG, 8.7m**
Incompatible Bids (block(s), bid)	B, 0.308m	CDE, 2.1m	CDE, 0.010m
	C, 0		
	E, 0		
	F, 0.310m		
VCG payment	**0**	**0**	**0**

The mechanics of the VCG payment calculations for this example are the same as in the example of the Introduction. U.S. Cellular’s bid of $8.7 million for blocks EFG prevented Dish Network from winning block G. Independently, T-Mobile’s bid of $3.5 million for blocks BCD also prevented Dish Network from winning block G because winning it would have forced U.S. Cellular to a noncontiguous assignment. Therefore, in the absence of either T-Mobile or U.S. Cellular, Dish Network would still have been assigned block A. As a result, the VCG payments of both T-Mobile and U.S Cellular equaled zero! In contrast, a core-selecting rule would have increased the joint payment of T-Mobile and U.S. Cellular to $237,000 and avoided the zero-revenue scenario.

Given that the problem of non-core outcomes was well known, the reader may wonder why the FCC adopted a VCG rule instead of a core-selecting rule for the assignment stage. The explanation is specific to this auction. In order for spectrum to be cleared in the Broadcast Incentive Auction, the revenues from the forward auction were required to exceed the costs from the reverse auction. This balanced budget constraint was based solely on the first, allocation stage and could not include revenues from the second, assignment stage (which had not yet occurred). Any higher payment rule in the assignment stage could induce lower bids and, as a result, lower revenues in the allocation stage. In turn, any diversion of revenues from the allocation stage created the risk of clearing less spectrum. To mitigate this risk, the lowest coherent payment rule for the assignment stage was adopted.

In the remaining sections, we will attempt to explain why the assignment stage is ripe for VCG outcomes outside the core.

4. A Two-Stage Vickrey Auction

An auctioneer seeks to auction m indivisible blocks of spectrum denoted by set $M = \{1, \ldots, m\}$. The set of all possible combinations of blocks (i.e., packages) is denoted by $\Omega = \{z \subseteq M\}$. The auctioneer intends to allocate these blocks to a set of bidders $N = \{1, \ldots, n\}$. For each bidder $i \in N$, the bidder's preferences over all packages in Ω are characterized by a value function $v_i(\cdot)$. The value of the null package, $\varnothing$, is normalized to zero. For now, we informally assume that bidders view all blocks in M as being "almost" homogeneous: each bidder mostly cares about the quantity of blocks it wins and cares less about the composition. Later, we provide some formal definitions that are needed for our results. Assumptions about value functions and information structure are as follows:

(A1) *Pure Private Values:* Each bidder knows its own value for any package and this value is not affected by the values of other bidders;

(A2) *Quasilinear Utility:* Each bidder i's payoff from winning package z and making a payment of p is given by $v_i(z) - p$;

(A3) *Complete Information:* The value function $v_i(\cdot)$ of each bidder i is known to all bidders in N.

The auctioneer can have additional restrictions on winnable packages. For example, the auctioneer can limit the number of blocks that can be won by an individual bidder (e.g., spectrum caps), or restrict multi-block winnings to form a contiguous block of spectrum. All such restrictions (if imposed by the auctioneer) are assumed to be captured by a set of feasible packages $\Omega \subseteq \Omega$.

The set of all feasible assignments is given by:

$$X = \left\{ (x_1, \ldots, x_n) : \; x_i \in \Omega \quad \text{and} \quad x_i \cap x_j = \varnothing \quad \forall i, j \in N : i \neq j \right\}, \tag{4.1}$$

and the coalitional value function for bidders in coalition $C \subseteq N$ is given by:

$$w(C) = \max_{x \in X} \sum_{j \in C} v_j(x_j) \; . \tag{4.2}$$

Assignment $x = (x_1, \ldots, x_n)$ is efficient if

$$\sum_{j \in N} v_j(x_j) = w(N). \tag{4.3}$$

A Vickrey outcome consists of an efficient assignment $x = (x_1, \ldots, x_n)$ and a payment vector $p^V = (p_1^V, \ldots, p_n^V)$ where

$$p_i^V \;=\; w(N\setminus\{i\}) \;-\; \sum_{N\setminus\{i\}} v_j(x_j), \tag{4.4}$$

and a core outcome consists of an efficient assignment $x = (x_1,\dots,x_n)$ and a payment vector $p^C = (p_1^C,\dots,p_n^C)$ belonging to the set of core payments $CP(x)$ given by:

$$CP(x) = \left\{ p^C \in \mathbb{R}^n : \; w(C) - \sum_C v_j(x_j) \;\le\; \sum_{N\setminus C} p_j^C \;\le\; \sum_{N\setminus C} v_j(x_j) \quad \forall C \subseteq N \right\}. \tag{4.5}$$

Both the Vickrey outcome and core outcomes with respect to bids are defined in the same way by substituting bids for values.

In this paper, we consider a two-stage procedure that uses a sequence of two Vickrey auctions. In the first stage (allocation stage), all blocks in M are treated as a homogeneous good. The auctioneer uses the Vickrey auction to determine the winning allocation of generic blocks and calculate corresponding payments.[11] In the second stage (assignment stage), the winners of the first stage compete for a specific assignment of generic blocks won in the first stage. Bidding in the second stage is optional since all winners of the first stage are necessarily assigned some blocks corresponding to their generic winnings. Once again, winning assignments and additional payments are determined by standard rules of the Vickrey auction.

Formally, in the allocation stage, each bidder i submits bid $b_i^I(q)$ for each quantity $q = 1,\dots,m$. Denote Y the set of all feasible quantity allocations, i.e.,

$$Y = \left\{ (y_1,\dots,y_n) : \; y_i \in \{0,\dots,m\} \quad \forall i \in N \quad \text{and} \quad \sum_{j\in N} y_j \le m \right\}. \tag{4.6}$$

The auctioneer identifies the winning allocation of generic blocks y^* by solving the following optimization problem:

$$y^* \in Y^* = \arg\max \sum_{j\in N} b_j^I(y_j) \qquad \text{s.t.} \qquad y \in Y. \tag{4.7}$$

If Y^* contains multiple solutions, the auctioneer selects y^* according to a pre-determined tie-breaking protocol.[12] Denote $\Omega(q)$ the set of all assignment options that are consistent with quantity q, i.e.,

$$\Omega(q) = \left\{ (z_1,\dots,z_m) \in \Omega : \; \sum_{k=1}^{m} z_k = q \right\}. \tag{4.8}$$

[11] The first stage could, instead, use a different auction format than the VCG mechanism, e.g., a uniform-price ascending clock auction. However, our analysis uses VCG for each stage of the auction, since it seems to give the fairest comparison (between the two-stage process and either a one-stage process or the efficient outcome).

[12] The tie-breaking procedure can have random components, but all random numbers must be drawn before the auction and disclosed to bidders such that they can perfectly predict an outcome in any tie-breaking situation.

In the assignment stage, each bidder i submits bid $b_i^{II}(z)$ for each assignment option $z \in \Omega(y_i^*)$. Denote $X(y)$ the set of all feasible assignments that correspond to the quantity allocation y, i.e.,

$$X(y) = \left\{ (x_1,...,x_n) : \; x_i \in \Omega(y_i) \quad \text{and} \quad x_i \cap x_j = \varnothing \;\; \forall i, j \in N : i \neq j \right\}. \tag{4.9}$$

The auctioneer selects the winning assignment x^* by solving the following optimization problem:

$$x^* \in X^* = \arg\max \sum_{j \in N} b_j^{II}(x_j) \qquad \text{s.t.} \quad x \in X(y^*). \tag{4.10}$$

Similar to (4.7), if X^* contains multiple solutions, the auctioneer selects x^* according to a pre-determined tie-breaking protocol. As an outcome of two stages, bidder i receives package x_i^* and pays the sum of two Vickrey payments, one from the allocation stage and another one from the assignment stage.

Truthful Bidding and Truthful Subgame Perfect Equilibrium

Next we define a notion of truthful bidding in the two-stage Vickrey auction. For each bidder i, denote $\underline{v}_i(q)$ and $\overline{v}_i(q)$ the values of the least preferred and the most preferred assignment options associated with q generic blocks, i.e.,

$$\underline{v}_i(q) = \min_{z \in \Omega(q)} v_i(z) \qquad \text{and} \qquad \overline{v}_i(q) = \max_{z \in \Omega(q)} v_i(z). \tag{4.11}$$

In the second stage, the Vickrey payment rule incentivizes bidders to bid truthfully on all feasible assignment options. For any given allocation of generic blocks $y = (y_1,..., y_n)$, truthful bidding for bidder i consists of bidding zero for its least preferred assignment option (since the bidder is guaranteed some assignment) and bidding its true marginal value on any other available option, i.e.,

$$b_i^{II}(z) = v_i(z) - \underline{v}_i(y_i) \qquad \forall z \in \Omega(y_i). \tag{4.12}$$

For each quantity allocation $y \in Y$, let $x_i(y)$ and $p_i(y)$ denote the assignment and payment, respectively, of bidder i in the second-stage Vickrey auction when all bidders bid truthfully according to (4.12).

In the first stage, the use of the Vickrey payment rule also incentivizes bidders to bid truthfully. However, the value of winning q generic blocks depends on the ultimate assignment a bidder would receive and the additional payment the bidder would have to make to get it in the second stage. By standard arguments, it follows that a bid $b_i^I(q)$ for q generic blocks is weakly dominated unless $b_i^I(q) \in [\, \underline{v}_i(q), \; \overline{v}_i(q) \,]$.

Denote $Y_i(q) = \{ y \in Y : \; y_i = q \}$ the set of all feasible allocations in which bidder i allocated q generic blocks, and let $\mu_i(q) \in Y_i(q)$ denote the belief of bidder i about the generic winnings profile after the first stage in which bidder i wins q generic blocks. Bidder i is said to bid truthfully in the first stage if it bids its "true" value of winning q generic blocks that incorporates the outcome of the assignment stage that would take place according to its belief $\mu_i(q)$, i.e.:[13]

$$b_i^I(q) = v_i(x_i(y)) - p_i(y) \qquad \text{where} \qquad y = \mu_i(q) \qquad \forall q = 1, \ldots, m. \tag{4.13}$$

In equilibrium, bidders' beliefs about which assignment rounds take place must be consistent with the first-stage bids.

Definition. A bid-belief profile $\left\{ b_i^I(\cdot), b_i^{II}(\cdot), \mu_i(\cdot) \right\}_{i=1}^n$ forms a *truthful subgame perfect equilibrium (TSPE)* of the two-stage Vickrey auction if

a) for any quantity allocation $y \in Y$ after the first stage, each bidder i bids truthfully in the second stage (according to (4.12));
b) based on its belief $\mu_i(\cdot)$, each bidder i bids truthfully in the first stage (according to (4.13));
c) belief $\mu_i(\cdot)$ for each bidder i is consistent given the first-stage bids of its opponents, i.e.,

$$\mu_i(q) = y^*(y_i = q) \qquad \forall q = 1, \ldots, m, \tag{4.14}$$

where $y^*(y_i = q)$ is the solution to the optimization problem (4.7) with an additional constraint restricting $y_i = q$.

We illustrate the TSPE concept with an example presented in Table 2. In this example, there are two blocks (A and B) and two bidders (1 and 2). In Stage 1, both bidders bid their true values for two generic blocks since winning two blocks unambiguously means winning blocks A and B. Deciding what to bid for one generic block is more involved since each bidder has to predict which block, A or B, it will ultimately get and at what price. When both bidders win one block each, Bidder 1 predicts that it will get block A and pay 5 in the assignment stage, and Bidder 2 predicts that it will get block B and pay 0. As a result, Bidder 1 bids $b_1^I(1) = v_1(A) - 5 = 40$ and Bidder 2 bids $b_2^I(1) = v_2(B) - 0 = 25$.

Table 2: Example with Unique Truthful Subgame Perfect Equilibrium

	Bidder 1	*Bidder 2*
Values	$v_1(A) = 45,\ v_1(B) = 39$	$v_2(A) = 30,\ v_2(B) = 25$
	$v_1(AB) = 60$	$v_2(AB) = 50$
VCG Outcome	$(A,\ 25)$	$(B,\ 15)$

Unique TSPE of the Two-Stage Vickrey Auction

[13] In an incomplete information model, note that the outcome of the assignment stage would depend on two separate pieces of private information: (1) the exact profile of generic winnings after the first stage; and (2) the assignment bids that will be placed by all bidders. This dependence of the first-stage bids on the outcome of the assignment stage presents a major obstacle for analyzing this model under incomplete information. Should we add [here or in the main body] that we therefore analyze a complete-information model?

Beliefs	$\mu_1(1) = (1,1),\ \mu_1(2) = (2,0)$	$\mu_2(1) = (1,1),\ \mu_2(2) = (0,2)$
Bids in Stage 1	$b_1^I(1) = 40,\ b_1^I(2) = 60$	$b_2^I(1) = 25,\ b_2^I(2) = 50$
Outcome of Stage 1	$(y_1 = 1,\ 25)$	$(y_2 = 1,\ 20)$
Bids in Stage 2	$b_1^{II}(A) = 6$	$b_2^{II}(A) = 5$
TSPE Outcome	$(A,\ 30)$	$(B,\ 20)$

It follows that in the unique TSPE, Bidder 1 wins block A and pays 30 (25 + 5) in total, and Bidder 2 wins block B and pays 20 (20 + 0) in total. Observe that while the equilibrium assignment is efficient, the equilibrium revenue is higher than the revenue generated by the one-stage VCG mechanism. The revenue difference between two auction formats is not surprising. While they both use the Vickrey pricing principle, the counterfactual calculations performed by each format are distinct.

In the second example, presented in Table 3, there are two blocks (A and B) and three bidders with single-unit demands. Both Bidder 1 and 2 prefer block A to block B while Bidder 3 is indifferent. It is easy to verify that there are two TSPEs in this setting, one with the efficient assignment and another with an inefficient assignment. In contrast to the previous example, the revenues in both TSPEs are lower than the revenue of the VCG mechanism.

Table 3: Possibility of Multiple Truthful Subgame Perfect Equilibria

	Bidder 1	*Bidder 2*	*Bidder 3*
Values	$v_1(A) = 110$	$v_2(A) = 109$	$v_3(A) = 102$
	$v_1(B) = 100$	$v_2(B) = 100$	$v_3(B) = 102$
VCG Outcome	$(A,\ 109)$	$(\varnothing,\ 0)$	$(B,\ 100)$

Multiple TSPEs of the Two-Stage Vickrey Auction

TSPE 1 (efficient)	*Beliefs*	$\mu_1(1) = (1,0,1)$	$\mu_2(1) = (1,1,0)$	$\mu_3(1) = (1,0,1)$
	Bids in Stage 1	$b_1^I(1) = 110$	$b_2^I(1) = 100$	$b_3^I(1) = 102$
	Outcome of Stage 1	$(y_1 = 1,\ 100)$	$(y_2 = 0,\ 0)$	$(y_3 = 1,\ 100)$
	Bids in Stage 2	$b_1^{II}(A) = 10$		
	TSPE Outcome	$(A,\ 100)$	$(\varnothing,\ 0)$	$(B,\ 100)$
TSPE 2 (inefficient)	*Beliefs*	$\mu_1(1) = (1,1,0)$	$\mu_2(1) = (0,1,1)$	$\mu_3(1) = (0,1,1)$
	Bids in Stage 1	$b_1^I(1) = 101$	$b_2^I(1) = 109$	$b_3^I(1) = 102$
	Outcome of Stage 1	$(y_1 = 0,\ 0)$	$(y_2 = 1,\ 101)$	$(y_3 = 1,\ 101)$
	Bids in Stage 2		$b_2^{II}(A) = 9$	
	TSPE Outcome	$(\varnothing,\ 0)$	$(A,\ 101)$	$(B,\ 101)$

The second example illustrates our first result: the outcome of the two-stage Vickrey auction may be outside the core with respect to true preferences, even when the outcome of the (one-stage) VCG mechanism is in the core. In the current example, a coalition that includes Bidder 2 and the auctioneer blocks the outcome of the efficient TSPE (TSPE 1 in Table 3). Note that this outcome is obtained even though the outcomes of Vickrey auctions in both stages are in the core with respect to bids.

Unfortunately, the TSPE is subject to a standard assortment of equilibrium problems including non-existence, non-uniqueness and inefficiency. The culprit is rather straightforward: splitting the auction process into two sequential stages does not work well unless bidders' preferences exhibit sufficient degree of homogeneity over licenses such that allocation of generic blocks has the first-order importance over deciding on a specific assignment. Any two allocations of generic blocks can be ranked as follows:

Definition. Allocation $y \in Y$ *value-dominates* another allocation $y' \in Y$ if

$$\sum_{S(y,y')} \underline{v}_j(y_j) \;>\; \sum_{S(y,y')} \overline{v}_j(y'_j), \tag{4.15}$$

where $S(y,y') = \{\, j \in N :\ y_j \neq y'_j \,\}$.

Intuitively, allocation y *value-dominates* another allocation y' when the aggregate value of the least preferred assignment options corresponding to y is greater than the aggregate value of the most preferred assignment options corresponding to y', ignoring the values of bidders who are allocated the same number of generic blocks in both allocations. Provided that the same blocks can be assigned to bidders who won the identical number of generic blocks in both allocations, any assignment $x \in X(y')$ is necessarily inefficient when allocation y' is value-dominated by some other allocation.[14]

We make the following additional assumptions on the collection of value functions $\{\, v_i(\cdot) \,\}_{i=1}^{n}$:

(A4) For each bidder $i \in N$ and each quantity $q = 1,\ldots,m$, there exists allocation $\tilde{y} \in Y_i(q)$ that value-dominates all other allocations in $Y_i(q)$.

(A5) There exists allocation $y^* \in Y$ that value-dominates all other allocations in Y.

To motivate both assumptions, consider a reduced game in which all bidders are limited to play undominated strategies in both stages. In this game, assumption (A4) guarantees that each bidder can perfectly predict which assignment round (i.e., the exact profile of generic winnings) will take place after the bidder wins q generic blocks in the first stage. Consequently, there exists only one set of beliefs $\{\, \mu_i(\cdot) \,\}_{i=1}^{n}$ that is consistent with any collection of undominated first-stage bids, and truthful bidding based on these beliefs forms the unique TSPE of the two-stage Vickrey auction. Intuitively, assumption (A4) ensures that the only bid profile that survives two rounds of iterated elimination of weakly dominated strategies applied to the full game corresponds to the unique TSPE.[15] In this sense, the unique TSPE of the two-stage Vickrey auction is analogous to the unique dominant-strategy equilibrium

[14] The contiguity restriction is an example of a constraint that can prevent the auctioneer from being able to assign the same blocks to bidders who won the identical number of generic blocks in allocations y and y'.

[15] Add a footnote with the order independence comment

of the VCG mechanism. Similarly, assumption (A5) guarantees that any undominated bids submitted in stage 1 result in the efficient allocation of generic blocks.

Proposition 1. Given assumptions (A1)—(A4), a truthful subgame perfect equilibrium in the two-stage Vickrey auction exists and it is unique. When assumption (A5) is also satisfied, the equilibrium assignment is efficient.

Proof: To prove existence, we need to show existence of correct beliefs. Due to assumption (A4), for any $y \in Y_i(q)$ and for truthful bids under any beliefs $\{\mu_i(\cdot)\}_{i=1}^n$, we have:

$$\sum_{S(y,\tilde{y})} b_j^I(y_j) \leq \sum_{S(y,\tilde{y})} \bar{v}_j(y_j) < \sum_{S(y,\tilde{y})} \underline{v}_j(\tilde{y}_j) \leq \sum_{S(y,\tilde{y})} b_j^I(\tilde{y}_j), \quad (4.16)$$

implying that allocation $\tilde{y} \in Y_i(q)$ is the only allocation in $Y_i(q)$ that can be consistent with the first-stage bids. Thus, setting the belief of bidder i for quantity q to the corresponding $\tilde{y}$ (i.e., $\mu_i(q) = \tilde{y}$) proves existence and uniqueness of TSPE. Efficiency of the equilibrium assignment follows immediately from the definition of TSPE and assumption (A5). *QED.*

5. Non-Core outcomes in the Two-Stage Vickrey Auction

In this section, we demonstrate that dichotomizing the auction into two stages might by itself cause extreme forms of bidder complementarities even in the most well-behaved settings. We do it with two examples satisfying assumptions (A1)—(A5) to guarantee efficiency and uniqueness of the TSPE.

The well-known sufficient and "almost necessary" condition for the outcome of the VCG mechanism to be in the core is bidders' preferences satisfying the gross substitutes (GS). Somewhat surprisingly, the GS condition does not provide any core guarantees for the two-stage Vickrey auction, as illustrated by the example in Table 4. In this example, there are two blocks (A and B) and each bidder's preferences satisfy gross substitutes: Bidder 1 has additive values over two blocks while Bidders 2 and 3 have single-unit demands. In the unique TSPE, the outcome of the first stage is outside the core with respect to bids: Bidders 2 and 3 jointly pay 10 for winning two generic blocks while Bidder 1 offered to pay 15. [16] Thus, the well-behaved preferences and truthful bidding do not guarantee that a core outcome would be observed in the two-stage Vickrey auction.

Table 4: A Non-core Outcome with GS preferences

	Bidder 1	*Bidder 2*	*Bidder 3*
Values	$v_1(A) = 5,\ v_1(B) = 10$	$v_2(A) = 15$	$v_3(A) = 16$
	$v_1(AB) = 15$	$v_2(B) = 15$	$v_3(B) = 22$

[16] One way to guarantee that the TSPE outcome of the first-stage is in the core with respect to bids is to require that bidders' marginal values are decreasing under any assignment. A bidder's preferences satisfy *assignment-independent decreasing marginal values (AIDMV)* if $\underline{v}_i(q) - \bar{v}_i(q-1) \geq \bar{v}_i(q+1) - \underline{v}_i(q)$ for any $q = 1, \ldots, m-1$. Both GS and AIDMV imply standard decreasing marginal values (i.e., the submodularity of a value function), however neither GS nor AIDMV implies each other.

VCG Outcome	$(\varnothing,\ 0)$	$(A,\ 5)$	$(B,\ 10)$

Unique TSPE of the Two-Stage Vickrey Auction

Beliefs	$\mu_1(1) = (1,0,1)$ $\mu_1(2) = (2,0,0)$	$\mu_2(1) = (0,1,1)$	$\mu_3(1) = (0,1,1)$
Bids in Stage 1	$b_1^I(1) = 5,\ b_1^I(2) = 15$	$b_2^I(1) = 15$	$b_3^I(1) = 22$
Outcome of Stage 1	$(y_1 = 0,\ 0)$	$(y_2 = 1,\ 5)$	$(y_3 = 1,\ 5)$
Bids in Stage 2			$b_3^{II}(B) = 6$
TSPE Outcome	$(\varnothing,\ 0)$	$(A,\ 5)$	$(B,\ 5)$

Notwithstanding the previous example, observe that the assignment stage preferences in spectrum settings are actually likely to violate the gross substitutes condition. For example, consider a bidder who wants two contiguous blocks (out of blocks A, B, C or D) and does not care which ones. If a bidder initially demands AB, its demand will switch to CD once the price of B has increased, causing a reduction for block A and violating substitutes. Such assignment complementarities are often "swamped" by bidders' preferences for quantities and cannot cause any core violations in the one-stage VCG mechanism. In contrast, same complementarities become focal once the assignment is determined in isolation under the two-stage approach. We illustrate this point with the example in Table 5.

Table 5: A Non-core Outcome in the Assignment Stage

	Bidder 1	*Bidder 2*	*Bidder 3*
Values*	$\underline{v}_1(1) = 150$ $v_1(A) = 170$ $\underline{v}_1(2) = 200$	$\underline{v}_2(1) = 100$ $v_2(B) = 120$ $\underline{v}_2(2) = 200$	$\underline{v}_3(1) = 300$ $\underline{v}_3(2) = 500$ $v_3(AB) = 520$
VCG Outcome	$(A,\ 100)$	$(B,\ 50)$	$(CD,\ 110)$

Unique TSPE of the Two-Stage Vickrey Auction

Beliefs	$\mu_1(1) = (1,1,2)$ $\mu_1(2) = (2,0,2)$	$\mu_2(1) = (1,1,2)$ $\mu_2(2) = (0,2,2)$	$\mu_3(1) = (1,2,1)$ $\mu_3(2) = (1,1,2)$
Bids in Stage 1	$b_1^I(1) = 170$ $b_1^I(2) = 200$	$b_2^I(1) = 120$ $b_2^I(2) = 200$	$b_3^I(1) = 300$ $b_3^I(2) = 500$
In-between Outcome	$(y_1 = 1,\ 80)$	$(y_2 = 1,\ 30)$	$(y_3 = 2,\ 110)$
Bids in Stage 2	$b_1^{II}(A) = 20$	$b_2^{II}(B) = 20$	$b_3^{II}(AB) = 20$
TSPE Outcome	$(A,\ 80)$	$(B,\ 30)$	$(CD,\ 110)$

* – for Bidder 1, value for A block is 170, value for any other block (B, C or D) is 150 and value for any 2 blocks is 200. Values for Bidders 2 and 3 are interpreted analogously.

In this example, there are four blocks (A, B, C and D) and three bidders with demand for at most two blocks. It is easy to verify that the one-stage VCG outcome is in the core, while the outcome of the unique TSPE of the two-stage Vickrey auction is not. In the first stage of the TSPE, Bidders 1 and 2 win one block each, and Bidder 3 wins 2 blocks. In the assignment stage, Bidder 1 bids 20 for A, Bidder 2 bids 20 for B, and Bidder 3 bids 20 for AB. As a result, Bidders 1 and 2 are now complements since they are jointly bidding against Bidder 3. The result of this complementarity is quite dreary for the auctioneer and Bidder 3: Bidders 1 and 2 win their preferred assignments and pay zero—an obvious core violation given the bid of Bidder 3!

In practice, there are a number of reasons why observing non-core outcomes in the first stage is less likely. First, a typical spectrum auctioneer allocates generic blocks via some dynamic auction format with a pricing rule other than the VCG. Second, a non-core outcome requires that in the absence of two or more winners, other bidders are able to acquire significantly larger amounts of spectrum. However, most spectrum auctions prevent bidders from buying large amounts of spectrum for competition reasons (e.g., "spectrum caps"), reducing the likelihood of outcomes outside the core (for example, see example in Table 5 where bidders' demands are limited to two blocks). In contrast, assignment stages are routinely implemented as sealed-bid auctions with pricing rules based on the opportunity costs principle (either the VCG mechanism or a core-selecting auction such as "Nearest-VCG"). Since opportunity costs in the assignment stage are coming from obtaining "better blocks" rather than obtaining "more blocks", spectrum caps have no effect on the likelihood of non-core outcomes.

6. Conclusion

The theoretical possibility for the VCG mechanism to generate an outcome outside the core has been known for almost 20 years. One of the most extreme ramifications of this problem is the possibility of zero-revenue outcomes. That is, in a generally competitive auction where all bidders (winners and losers alike) make nontrivial bids, winners pay zero to secure their allocations. This counterintuitive anomaly violates fundamental auction principles and leads to a host of problematic issues. In this paper, we document zero-revenue outcomes in the field for the first time: in the assignment stage of the FCC's Broadcast Incentive Auction.

The majority of recent spectrum auctions for near substitutes allocate them in two sequential phases: an allocation stage followed by an assignment stage. In this paper, we introduce a stylized model of a two-stage Vickrey auction and solve for its truthful equilibrium. Our findings are twofold. First, revenues in the efficient equilibrium of the two-stage Vickrey auction are generally different from the revenues generated by the VCG mechanism. As a result, even when outcomes of both stages are in the core with respect to bids, the outcome of the two-stage Vickrey auction might be outside the core with respect to true preferences.

Second, we demonstrate that the outcome of the assignment stage can be outside the core with respect to bids in settings that exclude any complementarities among goods (e.g., preferences exhibiting gross substitutes). We also show how the two-stage approach shifts focus on minor assignment complementarities that do not show up under the one-stage approach. Thus, the mere decomposition of the auction into two sequential stages can be responsible for creating bidder complementarities and generating non-core outcomes with respect to truthful bids.

Just because a two-stage procedure potentially increases the likelihood of non-core outcomes and because we have observed non-core outcomes in the assignment stage does not mean that we should not use a two-stage procedure. Quite the opposite, there are numerous practical advantages of the two-stage approach, including simplified bidding and reduced computational burden. But it does suggest that there may be compelling reasons to use a mechanism other than VCG for the assignment stage—for example, a core-selecting mechanism. Indeed, the published rules for the FCC's upcoming 24 GHz auction, which largely follows the same rules as the forward part of the Broadcast Incentive Auction, switches from VCG to a core-selecting mechanism for the assignment stage.

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