An Efficient Ascending Auction for Private Valuations

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Abstract

Known dynamic implementations of the Vickrey-Clarke-Groves mechanism in general private-value auction settings utilize non-linear (not additively-separable over goods) and non-anonymous (bidder-specific) prices. The need for non-linear and non-anonymous prices — a complication that is often difficult to implement in practice — arises from limiting attention to elicitation processes based on demand queries (i.e., asking bidders to report their demands at posted prices). In this paper, we relax this restriction and allow the auctioneer to supplement demand queries with marginal value queries (i.e., requests to report value differences between pairs of commodity bundles) as needed. This added flexibility enables an iterative ascending auction design that achieves efficiency despite using linear and anonymous prices.

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1 Introduction

Auction design is tasked with developing market mechanisms for allocating scarce resources. In many applications, the primary goal for an auctioneer is *efficiency* — allocating resources to bidders with the highest intrinsic values. The famous Vickrey-Clarke-Groves (VCG) mechanism is essentially the unique mechanism that achieves efficiency in dominant strategies without requiring money transfers by losing bidders.[1] Furthermore, recent studies have shown that the VCG mechanism has some compelling properties in broader settings. For example, the VCG mechanism induces efficient ex-ante investments that predetermine bidders' values and can be the revenue-maximizing choice for auctioneers competing for bidders.[2]

Motivated by virtues of the VCG mechanism and general benefits of dynamic auctions, researchers devoted major efforts to designing dynamic implementations of the VCG mechanism in various settings.[3] The list of benefits associated with dynamic auctions includes avoiding disclosure of winning values (privacy preservation), reducing the total amount of revelation of values by focusing elicitation on relevant bundles (bidding complexity), reducing the winner's curse through auction feedback, providing opportunities to manage budget constraints and reducing bidders' cognitive burden for placing bids.[4]

In simple settings, like selling or buying a single item, the main elements of a dynamic VCG auction are well-understood (e.g., the English auction). For more general settings, the auction literature has repeatedly attempted to characterize the main attributes of efficient dynamic auctions. For example, Gul and Stacchetti (2000) started their paper by putting forward the following definition of a dynamic auction:

> *"A dynamic auction can be described as a rule for adjusting prices given the observed history of demand (i.e., bids) and a rule for terminating the price adjustment procedure and specifying an allocation (i.e., determining who gets the good(s) and at what price(s)). The English auction is also identified with the property that prices are non-decreasing. More specifically, the English auction is typically identified with the procedure of increasing*

[1]VCG is due to Vickrey (1961), Clarke (1971) and Groves (1973). For the uniqueness result, see Green and Laffont (1979) and Holmstrom (1979).

[2]See Hatfield, Kojima and Kominers (2017) and Jehiel and Lamy (2017).

[3] See Demange et al. (1986), Gul and Stacchetti (2000), Parkes and Ungar (2000 and 2002), Ausubel and Milgrom (2002), Bikhchandani and Ostroy (2002 and 2006), Ausubel (2004 and 2006), de Vries et al. (2007), Mishra and Parkes (2007) and Lamy (2012). A comprehensive survey of this literature is provided by Parkes (2006).

[4]Reduction in cognitive burden is captured by the notion of *obvious strategy-proofness* by Li (2017).

the prices as long as there is excess demand."

The task of formally defining the class of dynamic auctions is challenging since their benefits, such as improved privacy preservation or reduced bidding complexity, are defined informally. Not surprisingly, the definition from Gul and Stacchetti (2000) was found to be rather restrictive and was relaxed in subsequent work. Ausubel (2006) allowed the auctioneer to collect bidders' demands along multiple price paths, while de Vries et al. (2007) let the auctioneer to solicit demands along a non-linear (i.e., not additively-separable over goods) and non-anonymous (i.e., specific for every bidder) price path. While these relaxations present theoretical interest, they are hardly practical since they involve rather awkward bidding procedures.

Despite many attempts to characterize dynamic auctions, one element of the original definition from Gul and Stacchetti (2000) has not been challenged — the auctioneer is restricted to quoting prices and collecting bidders' demands at these prices (i.e., the auctioneer is limited to demand queries). In this paper, we investigate whether relaxing this restriction enables new practical auction designs.

The motivation for our inquiry comes from the field where several modern auction designs have already adopted flexible ways for eliciting bidders' preferences. The Combinatorial Clock Auction (CCA) has been recently utilized for many spectrum auctions worldwide (Ausubel and Baranov (2017)). This auction format uses standard demand queries during the initial phase of the auction, and it allows additional sealed bids in the last round to supplement the previously revealed information. In 2016, the Combinatorial Multi-Round Ascending Auction (CMRA) was used to allocate 1800 MHz spectrum in Denmark. The CMRA uses standard demand queries, but it also allows bidders to place additional bids in each round. While not yet implemented in practice, Baranov et al. (2017) develop a new elicitation procedure suitable for settings with increasing returns: the auctioneer quotes a price and ask bidders to list all quantities they are willing to buy instead of asking them to report their demand.

For the general setting with private values, an ascending price auction that implements the VCG outcome was proposed by Mishra and Parkes (2007). Their design uses demand queries and requires a non-linear and non-anonymous price path, a limitation that handicaps the design's practical appeal.

We adopt several insights from Mishra and Parkes (2007) as our starting point. Our first contribution is a general class of iterative Vickrey auctions. We refer to an auction as iterative if it uses an iterative process to elicit bidders' preferences (i.e., bidders reveal their preferences in a step-by-step manner).[5] The class of iterative Vickrey auctions generalizes the class of ascending price auctions

[5] We intentionally distinguish between iterative auctions and dynamic auctions to avoid

introduced by Mishra and Parkes (2007) since it does not limit the auctioneer to a particular elicitation process. Instead, our class admits any elicitation process as long as it satisfies two technical properties. We prove that any auction in this class implements the VCG outcome as an ex-post equilibrium.[6] Thus, the problem of designing an iterative Vickrey auction can be simplified to designing an appropriate elicitation process that can take full advantage of application-specific details and objectives (e.g., relying only on linear and anonymous prices for preference elicitation).

Our main contribution is a characterization of an iterative ascending auction for general private valuations that implements the VCG outcome as an ex-post equilibrium while using a single linear and anonymous price path for elicitation. The insufficiency of linear and anonymous prices is circumvented by allowing the auctioneer to make additional marginal value queries in situations where standard demand queries miss critical information.[7]

We illustrate our approach with the following example. Consider a homogeneous good setting with three identical items and a bidder with values $v(1) = 20$, $v(2) = 25$, $v(3) = 40$. Suppose that the auctioneer quotes a per unit price p and asks the bidder to report its demand. Then the bidder reports 3 when $p < 10$ and 1 when $p \geq 10$. As a result, the auctioneer has no way to elicit $v(2)$ since the bidder never demands this quantity and cannot implement the VCG outcome in case it requires information about $v(2)$. Now suppose that the auctioneer also asks the bidder to report its marginal value for any "missed" quantity when its demand drops by more than one unit. Then, when bidder's demand drops from 3 to 1 at $p = 10$, the bidder would report that its marginal value for 2 units is $mv(2) = v(2) - v(1) = 5$. We show that this simple addition to the auctioneer's elicitation capabilities suffices to produce an iterative Vickrey auction for general private-value settings despite using linear and anonymous prices.

The paper is organized as follows. Section 2 presents the model, and Section 3 introduces a general class of iterative Vickrey auctions. The iterative ascending auction and main results are presented in Section 4. Section 5 discusses several implementation issues and Section 6 concludes. A discrete-price implementation

dealing with the problem of formally defining dynamic auctions. A dynamic auction has to be iterative since bidders must have multiple opportunities to communicate their preferences, but iterative auctions that do not have any benefits associated with dynamic auctions should not be classified as dynamic.

[6] This is a common result that appears in Gul and Stacchetti (2000), Ausubel (2004 and 2006), Bikhchandani and Ostroy (2006), de Vries et al. (2007) and Mishra and Parkes (2007) for their respective settings and auction designs. It is also related to results in the literature on preference elicitation in combinatorial auctions (see Conen and Sandholm (2001); for a comprehensive survey of this literature, see Sandholm and Boutilier (2006)).

[7] For the insufficiency of linear and anonymous prices, see Gul and Stacchetti (2000), Bikhchandani and Ostroy (2002), and Mishra and Parkes (2007).

of the auction and technical proofs can be found in Appendices A and B.

2 Model

A seller offers multiple units of K heterogeneous indivisible goods, denoted by vector $S = \{s^1, ..., s^K\} \in Z^K_{++}$, to a set of bidders $N = \{1, ..., n\}$. The set of all possible bundles of goods is denoted by $\Omega = \{(z^1, ..., z^K) : 0 \leq z^k \leq s^k \quad \forall\, k = 1, ..., K\}$. For every bidder $i \in N$, and every bundle $z \in \Omega$, the valuation of bidder i is given by $v_i(z)$, and the bidder's value for the null bundle $\emptyset$ is normalized to zero. We make the following standard assumptions:

(A1) *Pure Private Values:* Each bidder i knows its own valuation for any bundle z, and this valuation does not depend on valuations of other bidders;

(A2) *Quasilinear Values:* The payoff of bidder i from winning bundle z in exchange for a payment y is given by $v_i(z) - y$;

Another standard assumption made in the literature is monotonicity of value functions (i.e., free disposal). This assumption is not needed for our results.

An allocation $x = (x_1, ..., x_n)$ is called *feasible* if $x_i \in \Omega$ for all $i \in N$ and $\sum_N x_j \leq S$. The set of all feasible allocations is denoted by X.

We denote $E(M)$ an economy that only includes bidders in $M \subseteq N$. The *coalitional value function* for bidders in coalition $M \subseteq N$ is given by:

$$w(M) = \max_{x \in X} \sum_M v_j(x_j). \tag{2.1}$$

A feasible allocation $x = (x_1, ..., x_n) \in X$ is *efficient(optimal)* for economy $E(M)$ if

$$\sum_M v_j(x_j) = w(M). \tag{2.2}$$

A *Vickrey outcome* consists of an efficient allocation $x^* = (x^*_1, ..., x^*_n)$ for the main economy and a corresponding payment vector $y^V = (y^V_1, ..., y^V_n)$ such that

$$y^V_i = w(N_{-i}) - \sum_{N_{-i}} v_j(x^*_j) \qquad \text{for all} \quad i \in N, \tag{2.3}$$

where N_{-i} denotes the coalition of all bidders in N excluding bidder i.

3 Iterative Vickrey Auctions

In this section, we describe a general class of iterative auctions that implement the Vickrey outcome.

3.1 Preliminaries

An auction must obtain sufficient amount of information about bidders' values to prove the efficiency of a candidate allocation. The essential part of this task is a process for preference elicitation. Most elicitation processes in practice are designed after the famous *"Walrasian auctioneer"* by Walras (1874) — the auctioneer quotes prices and asks bidders to report their demand at these prices (i.e., demand queries). However, demand queries can be too limiting in some settings and the auctioneer must use other types of queries to elicit all necessary information.

A general framework for an iterative auction can be described as follows. At each time $t \geq 0$ (where $t = 0$ is the starting time), the auctioneer asks bidders to provide some information about their preferences and for each bidder i constructs an approximation of its value function $\hat{v}_i(., t)$ using bidder i's responses received on the time interval $[0, t]$. Here we assume that the auctioneer always generates a unique $\hat{v}_i(z, t)$ for each bundle $z \in \Omega$, and that $\hat{v}_i(\emptyset, t) = 0$ at any time t.

For each bidder i, denote the approximation error for bundle z at time t as

$$\delta_i(z, t) \;=\; v_i(z) \;-\; \hat{v}_i(z, t), \tag{3.1}$$

and let $\Delta_i(t)$ denote the set that contains all bundles with the highest approximation error at time t, i.e.,

$$\Delta_i(t) \;=\; \arg\max_{z \in \Omega} \delta_i(z, t). \tag{3.2}$$

By construction, $\delta_i(z, t) \geq 0$ for any bundle $z \in \Delta_i(t)$ since $\emptyset \in \Omega$. Note that set $\Delta_i(t)$ contains bundles for which true marginal values relative to each other can be recovered using $\hat{v}_i(., t)$ instead of the true value function $v_i(.)$.

Using the current approximations of value functions, the auctioneer finds $x^*(M, t)$, a tentative value-maximizing allocation for economy $E(M)$ at time t, by solving the following winner determination problem:

$$x^*(M, t) \;\in\; X^*(M, t) \;=\; \arg\max_{x \in X} \textstyle\sum_M \hat{v}_j(x_j, t) \tag{3.3}$$

If allocation $x^*(M, t)$ is not unique, the auctioneer can pick one of the allocations from $X^*(M, t)$ using some criteria.

Proposition 1 below is a partial restatement of the result obtained by Parkes (2002) and Nisan and Segal (2006) using our notation.[8] It provides a sufficient condition for proving the efficiency of a tentative allocation $x^*(M, t)$ for economy $E(M)$.

[8] Parkes (2002) and Nisan and Segal (2006) provide the necessary and sufficient conditions. In our case, only the sufficiency part can be established. Note that Proposition 1 corresponds to the First Fundamental Welfare Theorem with non-linear and non-anonymous prices.

Proposition 1. *[Parkes (2002), Nisan and Segal (2006)] A tentative allocation* $x^*(M,t) = (x_1, ..., x_n)$ *is efficient for economy* $E(M)$ *if*

$$x_i \ \in \ \Delta_i(t) \qquad \forall i \in M. \tag{3.4}$$

Proposition 2, due to Parkes and Ungar (2002) and Lahaie and Parkes (2004), provides the necessary and sufficient condition for recovering the Vickrey outcome.

Proposition 2. *[Parkes and Ungar (2002), Lahaie and Parkes (2004)] Suppose that at time t, condition (3.4) is satisfied for the main economy* $E(N)$*. Then the Vickrey outcome can be identified from* $\{\hat{v}_j(.,t)\}_{j\in N}$ *if and only if condition (3.4) is also satisfied for all marginal economies* $\{E(N_{-1}), E(N_{-2}), ..., E(N_{-n}\})$ *at time t.*

Propositions 1 and 2 establish the important role of condition (3.4). The only way to recover the Vickrey outcome using approximations of value functions $\{\hat{v}_j(.,t)\}_{j\in N}$ instead of true value functions is to uncover enough information to satisfy condition (3.4) in all relevant economies. However, this condition cannot be verified directly since set $\Delta_i(t)$ depends on true $v_i(.)$. Thus, to implement the Vickrey outcome, the auctioneer has to construct both an approximation of the value function $\hat{v}_i(.,t)$ and a feasible replacement for the set $\Delta_i(t)$ that can be used to validate condition (3.4).

3.2 Elicitation Process

In this section, we propose several properties for elicitation processes that are useful for designing iterative auctions. We make two technical assumptions. First, we assume that the auctioneer is limited to posing queries for which a well-defined truthful answer exists given bidder's true value function. Second, we limit our attention to fully expressive elicitation processes: an elicitation process is called *fully expressive* if the types of queries that the auctioneer is allowed to use are sufficient to elicit any value function that satisfies assumptions (A1) – (A2) from a bidder who responds truthfully.[9]

First, we provide a formal definition for an elicitation process. The definition highlights the equally important role of its two main elements: an approximation of the value function $\hat{v}_i(.,t)$ and a feasible replacement for set $\Delta_i(t)$, denoted by $\widehat{\Delta}_i(t)$.

Definition 1. *An* ***elicitation process*** *is a procedure that for each time* $t \geq 0$ *specifies queries addressed to each bidder i and converts bidder i's responses received on* $[0,t]$ *into a single-valued function* $\hat{v}_i(.,t)$ *and a set of bundles* $\widehat{\Delta}_i(t)$*.*

[9] The term "fully expressive" comes from the literature on bidding languages (see Nisan (2006)).

This definition is less innocent than it sounds. It implies that an elicitation process is sophisticated enough to resolve any inconsistencies in bidder i's responses to always produce a single-valued function $\hat{v}_i(., t)$ and set $\widehat{\Delta}_i(t)$.

The first property that we propose links an elicitation process with the true value function — all responses received from a truthful bidder should be treated as such; otherwise, a bidder wishing to communicate its true values would have to respond untruthfully.

Definition 2. *An elicitation process is called* ***straightforward*** *if it converts truthful responses of each bidder i made on the interval $[0, t]$ into a function $\hat{v}_i(., t)$ and set $\widehat{\Delta}_i(t)$ such that*

$$\widehat{\Delta}_i(t) \;\subseteq\; \Delta_i(t), \tag{3.5}$$

where $\Delta_i(t)$ is defined by (3.2) for $\hat{v}_i(., t)$ and true $v_i(.)$.

This property is a key for testing condition (3.4) as it allows the auctioneer to replace $\Delta_i(t)$ with $\widehat{\Delta}_i(t)$.

The second property that we propose requires that an elicitation process is capable, if needed, to fully elicit bidder i's value function, and the complete elicitation of $v_i(.)$ implies that $\Delta_i(t) = \Omega$. To guarantee this property, we require that an elicitation process weakly expands set $\widehat{\Delta}_i(t)$ over time. In other words, the auctioneer progressively builds up the set of bundles that can be awarded to bidder i while never discarding bundles that have been already added to $\widehat{\Delta}_i(t)$.

Definition 3. *An elicitation process is called* ***iterative*** *if the set $\widehat{\Delta}_i(t)$ for each bidder i is weakly increasing in time, i.e.,*

$$\widehat{\Delta}_i(t') \;\subseteq\; \widehat{\Delta}_i(t) \qquad \textit{for all} \quad t' \;\leq\; t. \tag{3.6}$$

Next, we define several monotonicity restrictions that can be imposed on an elicitation process. These restrictions are not required for our main results, but they are frequently used in practice to yield a monotonic discovery of an auction outcome.

One of the most popular restrictions utilized in practice is a restriction on a direction for adjustments to approximations of value functions. These restrictions ensure that the approximation monotonically approaches the true value function from below or from above.

Definition 4. *An iterative elicitation process is called* ***ascending (descending)*** *if for each bidder i and any bundle $z \in \widehat{\Delta}_i(t)$ ($z \notin \widehat{\Delta}_i(t)$), $\hat{v}_i(z, t)$ is non-decreasing (non-increasing) in time.*[10]

[10]This definition is weaker than the one provided by Mishra and Parkes (2007) where the direction for adjustments is constrained for all bundles $z \in \Omega$.

Another desirable form of monotonicity in iterative auctions is a guarantee that once a tentative allocation has been proven optimal for economy $E(M)$, it continues to be optimal in the future. We say that economy $E(M)$ is *cleared at time* t if there exists a tentative allocation $x^*(M,t) = (x_1, ..., x_n)$ that solves (3.3) such that

$$x_i \in \widehat{\Delta}_i(t) \qquad \forall i \in M. \tag{3.7}$$

It is easy to verify that the next property guarantees that a cleared economy stays cleared at a later time by ensuring that value adjustments for all bundles in $\widehat{\Delta}_i(t)$ are at least as high as for bundles outside of $\widehat{\Delta}_i(t)$.

Definition 5. *An iterative elicitation process is* ***monotonic*** *if for each bidder* i*, any time* $t' \leq t$ *and any pair of bundles* $z \in \Omega$ *and* $y \in \widehat{\Delta}_i(t')$*,* $\hat{v}_i$ *satisfies*

$$\hat{v}_i(z,t) - \hat{v}_i(z,t') \leq \hat{v}_i(y,t) - \hat{v}_i(y,t'). \tag{3.8}$$

3.3 Iterative Vickrey Auctions

In this section, we define a general class of iterative Vickrey auctions and show that they implement the Vickrey outcome as an ex-post equilibrium. First, we specify an adjustment rule that moves the elicitation process along by forcing bidders to successively reveal more competitive values. To facilitate greater generality, we only require that at any time at least one bidder is asked to revise its values to guarantee the termination of the auction in a finite time. The termination time is denoted T and specified later as part of the definition of an iterative Vickrey auction (see Definition 7).

Definition 6. *An iterative elicitation process satisfies the adjustment rule if there exist* $\epsilon > 0$ *and* $\lambda > 0$ *such that for any* t, t' *where* $t' + \lambda \leq t \leq T$*, there exists bidder* $i \in N$ *for whom either*

$$\hat{v}_i(z,t) - \hat{v}_i(z,t') \geq \epsilon \qquad \textit{for all} \quad z \in \widehat{\Delta}_i(t') \quad \textit{or} \tag{3.9}$$

$$\hat{v}_i(z,t') - \hat{v}_i(z,t) \geq \epsilon \qquad \textit{for some} \quad z \notin \widehat{\Delta}_i(t'). \tag{3.10}$$

Intuitively, condition (3.9) forces bidder i to increase values for all bundles in $\widehat{\Delta}_i(t')$, an adjustment typical for an ascending auction in which bidders increase the implicit values for their current demands (like in the English auction). Alternatively, condition (3.10) forces bidder i to decrease values for some bundles outside of $\widehat{\Delta}_i(t')$, an adjustment typical for a descending auction where the implicit values for bundles decrease until bidders start demanding them (like in the Dutch auction). Now we define a general class of iterative Vickrey auctions.

Definition 7. *An iterative Vickrey auction is an auction procedure that:*

(1) uses a straightforward and iterative elicitation process that satisfies the adjustment rule;

(2) terminates at the first time t such that all economies in $\{E(N), E(N_{-1}), ..., E(N_{-n})\}$ are cleared at the same time (termination time $T := t$);

(3) awards bundle $x_i^*(N,T)$ to bidder i in exchange for a payment

$$y_i^V \;=\; \sum_{j \,\in\, N_{-i}} \left[\hat{v}_j(x_j^*(N_{-i},T)) \;-\; \hat{v}_j(x_j^*(N,T))\right]. \tag{3.11}$$

Our class of iterative Vickrey auctions is intentionally very permissive due to a flexible adjustment rule. For example, the class admits iterative implementations that are essentially sealed-bid.[11] In practice, the auctioneer should use elicitation processes and adjustment rules that deliver advantages of dynamic auctions.

Theorem 1. *If each bidder i bids truthfully according to $v_i(.)$, an iterative Vickrey auction implements the Vickrey outcome, and truthful bidding by all bidders is an ex-post equilibrium.*

Theorem 1 generalizes the standard result in the literature that was previously established for specific elicitation processes and specific auction settings. More generally, this result allows us to simplify the problem of designing an iterative Vickrey auction to an easier problem of designing straightforward and iterative elicitation processes that are tailored to application-specific objectives such as speed, privacy, feedback, bidding convenience, and etc. In the next section, we adopt this approach to construct an iterative ascending auction for the general private-value setting that implements the Vickrey outcome and uses a single linear and anonymous ascending price path for preference elicitation.

4 Iterative Ascending Auction

This section contains our main results. We construct an efficient iterative ascending auction with an elicitation process that is driven by a single linear and anonymous price path.

[11] Consider an iterative auction where an auctioneer uncovers value functions sequentially (first, for bidder 1, then bidder 2, then bidder 3, and etc.) without providing feedback to bidders. This iterative auction is equivalent to a sealed-bid auction since it does not deliver any benefits associated with true dynamic auction designs.

4.1 Elicitation Process

The auctioneer initializes K clock prices, one for each good, at zero. At any time $t \geq 0$, the auctioneer announces current clock prices $p(t)$, and each bidder i replies with a single bundle $x_i(t) \in \Omega$ that is treated as bidder i's demand at current prices. The price trajectory $p(.)$ is assumed to be non-decreasing, continuous and piecewise linear on $[0, +\infty)$. The demand function $x_i(.)$ for each bidder i is assumed to be a right-continuous piecewise constant function.[12]

It is well known that demand queries in combination with a linear and anonymous price path do not produce a fully expressive elicitation process, so the auctioneer has to use different types of queries to elicit extra information when needed. With non-decreasing clock prices, a truthful bidder would never demand a bundle that is a superset of its demand at an earlier time due to the *"law of demand"* — a requirement that demand and prices move in opposite directions. Since values for such bundles cannot be elicited with any feasible demand queries, they must be recovered in some other way. One possible approach is to ask bidders to report their marginal values for such bundles relative to their current demand.

Denote $\widehat{\Delta}_i(t)$ the set of revealed bundles that includes any bundle $z \in \Omega$ that is a superset of bidder i's demand for some time $t' \in [0, t]$, i.e.,

$$\widehat{\Delta}_i(t) \;=\; \big\{ z \in \Omega :\; \exists\, t' \in [0, t] \;\text{ such that }\; z \,\geq\, x_i(t') \,\big\}. \tag{4.1}$$

By construction, the set of revealed bundles $\widehat{\Delta}_i(t)$ only expands when bidder i demands a new bundle at time t. At this time, bidder i is required to report its marginal values relative to its demand $x_i(t)$ for all bundles that are newly added to the set $\widehat{\Delta}_i(t)$. For any newly added bundle z, denote $\widetilde{mv}_i(z)$ the reported marginal value of bundle z relative to $x_i(t)$; and denote $t(z)$ the time when bundle z is added to $\widehat{\Delta}_i(t)$. Note that in some scenarios, the number of bundles that are added to the set of revealed bundles at one time can be exponential and lead to practical challenges (for example, if bidder i drops its demand from S to the null bundle $\emptyset$ at time t, then all bundles in Ω are added to $\widehat{\Delta}_i(t)$ at the same time). A detailed discussion of this issue can be found in Section 5.3.

We say that bidder i *bids truthfully according to its value function* $v_i(.)$ on $[0, t]$ if at any time $s \in [0, t]$:

(a) bidder i truthfully reports its demand $x_i(s)$ given $p(s)$, i.e.,

$$x_i(s) \;\in\; \underset{z \in \Omega}{\arg\max}\; \big[v_i(z) \;-\; p(s)\, z \big]; \quad \text{and} \tag{4.2}$$

[12] This assumption is without loss of generality. It can be shown that a piecewise constant demand function exists for any function $v(.)$ satisfying assumptions (A1) – (A2) provided that the price path $p(.)$ is continuous and piecewise linear on $[0, +\infty)$.

(b) bidder i truthfully reports its marginal value for all bundles that are added to the set of revealed bundles at time s, i.e., for any bundle z that is added to $\widehat{\Delta}_i(s)$ at time s,

$$\widetilde{mv}_i(z) \;=\; v_i(z) \;-\; v_i(x_i(s)). \tag{4.3}$$

To ensure that each bidder bids according to some value function, the elicitation process needs activity rules that are based on *the Generalized Axiom of Revealed Preference (GARP)*. The most famous result in the literature on revealed preferences is the *Afriat's Theorem* due to Afriat (1967). It establishes a direct connection between GARP and existence of a value function that rationalizes demand $x_i(.)$ given price path $p(.)$. Using our notation, the Afriat's theorem is stated as follows.

Afriat's Theorem (1967). *Given price path $p(.)$, bidder i's demand $x_i(.)$ is rationalized by a value function satisfying assumptions (A1) and (A2) if and only if its demand $x_i(.)$ satisfies GARP on $[0,t]$, i.e.,*[13]

$$p(s)[x_i(s) \;-\; x_i(s')] \;+\; \int_s^{s'} p(u)dx_i(u) \;\leq\; 0 \qquad \forall\; s, s' \in [0,t]. \tag{4.4}$$

Ausubel and Baranov (2018) introduced the following notion of violating GARP. Suppose that bidder i's demand satisfies GARP on interval $[0,t]$. Then we can hypothetically ask whether bidder i would have violated GARP by demanding bundle z instead of $x_i(t)$ at time t by calculating $gv_i(z,t)$ defined as

$$gv_i(t,z) \;=\; \max_{s\,\in\,[0,t]} \left\{ p(s)[x_i(s) - z] \;+\; \int_s^t p(u)dx_i(u) \;+\; p(t)[z - x_i(t)] \right\}. \tag{4.5}$$

Intuitively, $gv_i(t,z)$ is the maximum net amount that can be extracted from bidder i in a series of transactions that starts and ends with the same bundle z. By construction, $gv_i(t,z) \geq 0$ for all $z \in \Omega$ and $gv_i(t,z) > 0$ indicates that bidding for bundle z at time t violates rationality (i.e., proves the existence of a "money pump").[14] Furthermore, when bidder i bids truthfully according to $v_i(.)$, the upper bound on the marginal value of bundle z relative to $x_i(t)$ is given by

$$v_i(z) \;-\; v_i(x_i(t)) \;\leq\; p(t)\,[z \;-\; x_i(t)] \;-\; gv_i(t,z). \tag{4.6}$$

[13] The integral term in (4.4) and all similar formulas in the paper is a Stieltjes integral. A definition of the Stieltjes integral can be found in Apostol (1957, Definition 9-1). See Ausubel (2006, pp. 609–610) for a short discussion.

[14] For a piecewise linear price path $p(.)$ and piecewise constant demand $x_i(.)$, the optimization problem (4.5) has a linear programming formulation that can be used to calculate $gv_i(t,z)$ in applications.

For our purposes, we need two activity rules: one to restrict changes in demand; and another one to limit the reported marginal values for bundles that are added to the set of revealed bundles.

AR1: At time t, bundle $z \in \Omega$ is unacceptable as demand of bidder i if either $gv_i(t,z) > 0$ or $z \geq x_i(s)$ for some time $s < t$ (when validating AR1, $gv_i(t,z)$ is calculated using (4.5) by setting $x_i(t) := z$);

AR2: For bundle z that is added to $\widehat{\Delta}_i(t)$ at time t, the reported marginal value $\widetilde{mv}_i(z)$ has to satisfy the following inequality

$$\widetilde{mv}_i(z) \;\leq\; p(t)\,[z \,-\, x_i(t)] \;-\; gv_i(t,z). \tag{4.7}$$

AR1 forces bidder i to submit demands that can be rationalized by some value function and also precludes bids for any supersets of bundles that were demanded before. AR2 restricts the marginal value report by the highest possible value for bundle z that is consistent with the fact that bidder i has never demanded bundle z before.[15] Proposition 3 proves that a bidder must bid according to some value function when limited by these activity rules.[16]

Proposition 3. *Given non-decreasing price path $p(.)$, bidder i bids truthfully according to some value function on $[0,t]$ if and only if its bids are constrained by activity rules AR1 and AR2.*

Now we construct the approximation of value function $\hat{v}_i(.,t)$ to complement the set of revealed bundles $\widehat{\Delta}_i(t)$. For any bundle $z \in \widehat{\Delta}_i(t)$, denote its revealed marginal value relative to bidder i's current demand $x_i(t)$ as:

$$mv_i(z,t) \;=\; \begin{cases} \int\limits_t^{t'} p(u)dx_i(u) & \text{if } \exists\, t' \in [0,t] : \, x_i(t') \,=\, z \\ \int\limits_t^{t(z)} p(u)dx_i(u) \,+\, \widetilde{mv}_i(z) & \text{otherwise} \end{cases} \tag{4.8}$$

Intuitively, the marginal value for bundle z is elicited either by following the changes in demand if the bundle was previously demanded by bidder i, or its

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[15] For a homogeneous setting with $K = 1$, AR1 simplifies to demand monotonicity (i.e., $x_i(t) \leq x_i(s)$ for any $s \leq t$) and AR2 simplifies to $\widetilde{mv}_i(z) \leq p(t)\,[z - x_i(t)]$ (since any bundle z that is added to $\widehat{\Delta}_i(t)$ via a marginal value report at time t corresponds to a "missed" quantity, $x_i(t) < z < x_i(t-0)$, for which $gv_i(t,z) = 0$).

[16] Surprisingly, constraining reported marginal values with AR2 at time t turns out to be sufficient for the existence of a value function that rationalizes bidding at later times. To put it differently, with a non-decreasing price path, the upper bound for $\widetilde{mv}_i(z)$ from (4.7) derived at time t does not change later.

marginal value was elicited with an additional marginal value query at time $t(z)$ when bundle z was added to $\widehat{\Delta}_i(t)$. Using revealed marginal values, the auctioneer constructs the current approximation of the value function as follows:

$$\hat{v}_i(z,t) \;=\; \begin{cases} p(t)\,x_i(t) \;+\; mv_i(z,t) & \text{if} \quad z \in \widehat{\Delta}_i(t) \\ p(t)\,z \;-\; gv_i(t,z) & \text{if} \quad z \notin \widehat{\Delta}_i(t) \end{cases} \tag{4.9}$$

For any revealed bundle $z \in \widehat{\Delta}_i(t)$, formula (4.9) uses the revealed marginal value between bundle z and the current demand $x_i(t)$ imputing the current clock price for the current demand $x_i(t)$. For any non-revealed bundle $z \notin \widehat{\Delta}_i(t)$, the formula imputes the maximum possible value for bundle z that is consistent with the current bidding history of bidder i to ensure that these bundles create maximum competition for bundles in $\widehat{\Delta}_i(t)$. The latter part guarantees that the elicitation process is straightforward and monotonic.[17] Properties of the elicitation process are summarized by Proposition 4.

Proposition 4. *If bidder i is constrained by activity rules AR1 and AR2, then $\hat{v}_i(.,t)$ as defined in (4.9) and $\widehat{\Delta}_i(t)$ as defined in (4.1) constitute an elicitation process that is straightforward, iterative, ascending and monotonic.*

Clock auctions traditionally use the excess demand to adjust clock prices and determine whether the auction reached its end. In the next section, we define an appropriate notion of excess demand for our auction and propose a simple clock price adjustment process.

4.2 Excess Demand and Clock Increments

The last ingredient for our iterative ascending auction is a rule to adjust clock prices. Such rule has to be linked to the closing rule to guarantee that prices are incremented in a meaningful way until the closing rule is met. We adopt a standard approach of specifying a notion of excess demand with a property that the closing rule is satisfied if and only if there is no excess demand for any goods. When the closing rule is not satisfied, the excess demand is positive for at least one good and increasing the clock prices for goods with excess demand puts pressure on bidders to reveal more competitive values.[18]

[17] If the auctioneer imputes 0 for $z \notin \widehat{\Delta}_i(t)$, the resulting elicitation process is not straightforward, and imputing $p(t)\,z$ for $z \notin \widehat{\Delta}_i(t)$ leads to a straightforward elicitation process that is not monotonic.

[18] The classic notion of excess demand does not work in this environment since (1) it does not match the closing rule that has to account for marginal economies; and (2) there is a possibility that an efficient allocation does not allocate all goods to bidders.

By definition, economy $E(M)$ is cleared at time t if there exists a tentative assignment $x^*(M,t)$ that assigns each bidder in M a bundle from its corresponding set of revealed bundles $\widehat{\Delta}_i(t)$. The failure to clear economy $E(M)$ is traced to bidders whose tentative winnings are not in their sets of revealed bundles and motivates the following construction of the excess demand. When bidder i is assigned a bundle from $\widehat{\Delta}_i(t)$, bidder i does not prevent economy $E(M)$ from clearing, and its contribution towards the excess demand is zero. When bidder i is assigned a bundle outside of $\widehat{\Delta}_i(t)$, bidder i does prevent economy $E(M)$ from clearing, and its contribution towards the excess demand must account for its current demand for goods that are not awarded to the bidder in its tentative assignment. This construction has two sources of multiplicity. First, there can be multiple tentative value-maximizing allocations for economy $E(M)$, and second, some bidders might be implicitly demanding multiple bundles at $p(t)$.

Formally, define $D_i(t)$ the revealed demand correspondence of bidder i at time t as

$$D_i(t) \;=\; \underset{z\,\in\,\widehat{\Delta}_i(t)}{\arg\max}\;[\,\hat{v}_i(z,t)\;-\;p(t)\,z\,]. \tag{4.10}$$

The set of excess demands *(excess demand correspondence)* for economy $E(M)$ at time t is defined as

$$Z(M,t) \;=\; \{\; \textstyle\sum_M z_j \;\} \tag{4.11}$$

such that there exists allocation $x^* \in X^*(M,t)$ and for each bidder $i \in M$, $z_i \;=\; 0$ if $x_i^* \in \widehat{\Delta}_i(t)$ and $z_i \;=\; \max\{\,0,\, d \,-\, x_i^*\,\}$ if $x_i^* \notin \widehat{\Delta}_i(t)$ where $d \in D_i(t)$.

The next proposition shows that the clearing of economy $E(M)$ at time t is equivalent to a familiar "zero excess demand" condition.

Proposition 5. *If all bidders are constrained by activity rules AR1 and AR2, economy $E(M)$ is cleared at time t if and only if* $0 \in Z(M,t)$.

When economy $E(M)$ is not cleared at time t, any excess demand vector $z \in Z(M,t)$ is positive for at least one good. The auctioneer can direct the auction process towards clearing economy $E(M)$ by picking one of the excess demand vectors from $Z(M,t)$ and increasing clock prices for goods with excess demand. While there are many potential ways to pick one of the excess demand vectors from $Z(M,t)$, the simplest approach is to choose the one that minimizes the current price value of excess demand, i.e.,

$$z(M,t) \;\in\; \underset{z\,\in\,Z(M,t)}{\arg\min}\;\; p(t)\,z\;. \tag{4.12}$$

To recover the Vickrey outcome, the auctioneer needs to clear the main economy and all marginal economies. An important consideration for the auctioneer is the order in which these economies are targeted. For ascending auctions, it is natural for marginal economies to clear ahead of the main economy due to weaker competition. Clearing marginal economies before the main economy is a desirable property since an unnatural order can lead to incentive problems (see Section 5.2 for a discussion of this issue).

For simplicity, we assume that the auctioneer simultaneously targets all relevant economies. For each good $k \in \{1, ..., K\}$, define a "cumulative" excess demand at time t as the maximum of excess demands for the good among all relevant economies $M \in \{N, N_{-1}, ..., N_{-n}\}$, i.e.,

$$Z^k(t) \quad = \max_{M \in \{N, N_{-1}, ..., N_{-n}\}} z^k(M, t). \tag{4.13}$$

A naive price adjustment process based on the excess demand (which in turn depends on individual demands) can cause a known technical problem with infinitive price oscillations (see Gul and Stacchetti (2000) and Ausubel (2006)). To avoid this problem and to provide a way for a practical implementation, we adopt a price adjustment process that regulates the speed of price clocks at discrete times. This approach is known as *intra-round bidding* and to the best of our knowledge, is the only way of implementing continuous price clocks in practice.

Formally, the auctioneer initializes price clocks at zero $p(0) = 0$ and asks for initial reports to construct $\widehat{\Delta}_i(0)$ for each bidder $i \in N$. Then, at any time $t \geq 0$, the auctioneer sets the clock price for good $k \in \{1, ..., K\}$ using

$$p^k(t) \;=\; \begin{cases} p^k(t')\,[1 \;+\; \epsilon\,(t \;-\; t')] & \text{if} \quad Z^k(t') \;>\; 0 \\ p^k(t') & \text{if} \quad Z^k(t') \;=\; 0 \end{cases}, \tag{4.14}$$

where t' is the highest integer such that $t' < t$ and $\epsilon > 0$. Intuitively, the price adjustment in (4.14) uses the excess demand at integer time t' to determine which clock prices will be increased on the time interval $(t', t' + 1]$ and then increases them at a constant speed ϵ (for example, $\epsilon = 0.05$ corresponds to a 5% increment on the time interval $[t', t' + 1]$ for goods with positive excess demand). This price adjustment process produces a continuous piecewise linear non-decreasing price path.

It is important to emphasize that the price adjustment process (specified by formulas (4.10) – (4.14)) is just one possible alternative. Designing price adjustment processes that offer some advantages in terms of reducing the number of queries in various settings is an interesting direction for future research.

4.3 Iterative Ascending Auction

In this section, we specify our iterative ascending auction and prove that it implements the Vickrey outcome as an ex-post equilibrium (Theorem 2).

Iterative Ascending Auction: The iterative ascending auction consists of the following components:

(1) The auctioneer initializes clock prices at zero $p(0) = 0$. At each time $t \geq 0$, the auctioneer quotes clock prices $p(t)$ and asks each bidder i to report (1) its demand $x_i(t)$; and (2) its marginal value for any bundle that is added to the set of revealed bundles $\widehat{\Delta}_i(t)$ at time t. All responses are subject to activity rules AR1 and AR2;

(2) At each time $t \geq 0$ and for each bidder i, the auctioneer constructs $\widehat{\Delta}_i(t)$ and $\hat{v}_i(., t)$ using formulas (4.1) and (4.9). In addition, the auctioneer calculates excess demand $Z(t)$ according to formula (4.13);

(3) If $Z(t) \neq 0$ at time t, the clock prices are adjusted using the price adjustment process (4.14) and the process goes back to step (2). If $Z(t) = 0$, then the process terminates ($T := t$) and bidder i is awarded bundle $x_i^*(N, T)$ in exchange for a payment

$$y_i^V \ = \ \sum_{j \in N_{-i}} \left[\hat{v}_j(x_j^*(N_{-i}, T)) \ - \ \hat{v}_j(x_j^*(N, T)) \right]. \tag{4.15}$$

Theorem 2. *If each bidder i bids truthfully according to $v_i(.)$, the iterative ascending auction implements the Vickrey outcome, and truthful bidding by all bidders is an ex-post equilibrium.*

4.4 An Illustrative Example

To illustrate the iterative ascending auction, we use an example with two unique goods, A and B, and three bidders. Bidder 1 values good A at 3 and has no value for good B. Bidder 2 values good B at 10 and has no value for good A. Bidder 3 values bundle AB at 8, and her value for any standalone good, either A or B, is only 1. The auctioneer initializes price clocks at $p(0) = (0, 0)$ and increments the clock price for any good with excess demand at a constant speed. For convenience, we assume that all bidders demand the smallest bundle when indifferent. All necessary details for this example are provided in Table 1.

In this example, the development of the iterative ascending auction can be decomposed into three stages. In stage one, clock prices rise from $p(0) = (0, 0)$ to $p(3) = (3, 3)$. At $p(3)$, Bidder 1 stops demanding good A and drops to $\emptyset$ which allows the auctioneer to elicit Bidder 1's value for A and AB. However, the value for good B is still missing, and the auctioneer asks Bidder 1 to report its marginal value for B relative to its current demand.

Table 1: An Illustrative Example of the Iterative Ascending Auction

	Bidder 1	*Bidder 2*	*Bidder 3*		
Values (A, B, AB):	$v_1 = (3, 0, 3)$	$v_2 = (0, 10, 10)$	$v_3 = (1, 1, 8)$		
Efficient Allocation:	A	B	$\emptyset$		
Vickrey Payments:	1	5	0		
Clock Prices		*Bidding Information*		$Z(t)$	$Z(N, t)$
$t: 0 \to 3$ $p(t) = (t, t)$	$x_1 = A$ $\widehat{\Delta}_1 = \{A, AB\}$ $\hat{v}_1 = (t, t, t)$	$x_2 = B$ $\widehat{\Delta}_2 = \{B, AB\}$ $\hat{v}_2 = (t, t, t)$	$x_3 = AB$ $\widehat{\Delta}_3 = \{AB\}$ $\hat{v}_3 = (t, t, 2t)$	(1, 1)	(1, 1)
$t: 3 \to 5$ $p(t) = (3, t)$	$x_1 = \emptyset$ $\widehat{\Delta}_1 = \Omega$ $\hat{v}_1 = (3, 0, 3)$	$x_2 = B$ $\widehat{\Delta}_2 = \{B, AB\}$ $\hat{v}_2 = (3, t, t)$	$x_3 = AB$ $\widehat{\Delta}_3 = \{AB\}$ $\hat{v}_3 = (3, t, 3 + t)$	(0, 1)	(0, 1)
$t: 5 \to 7$ $p(t) = (3, t)$	$x_1 = \emptyset$ $\widehat{\Delta}_1 = \Omega$ $\hat{v}_1 = (3, 0, 3)$	$x_2 = B$ $\widehat{\Delta}_2 = \{B, AB\}$ $\hat{v}_2 = (3, t, t)$	$x_3 = \emptyset$ $\widehat{\Delta}_3 = \Omega$ $\hat{v}_3 = (1, 1, 8)$	(0, 1)	(0, 0)
At termination: $T = 7$ $p(T) = (3, 7)$	$x_1 = \emptyset$ $\widehat{\Delta}_1 = \Omega$ $\hat{v}_1 = (3, 0, 3)$	$x_2 = B$ $\widehat{\Delta}_2 = \{B, AB\}$ $\hat{v}_2 = (3, 7, 7)$	$x_3 = \emptyset$ $\widehat{\Delta}_3 = \Omega$ $\hat{v}_3 = (1, 1, 8)$	(0, 0)	(0, 0)

Notes: Given the bidding history, for any time $t \in [0, T]$ and any bidder $i \in N$, $gv_i(t, z) = 0$ for any $z \notin \widehat{\Delta}_i(t)$ (i.e., for any time t and each bidder i, it would be consistent with AR1 to demand bundle $z \notin \widehat{\Delta}_i(t)$ instead of $x_i(t)$).

At $p(3)$, there is no excess demand for good A and only the clock price for good B is rising in stage two. At $p(5) = (3, 5)$, Bidder 3 reduces its demand to $\emptyset$. At this time, the auctioneer knows that the value of Bidder 3 for AB is 8, but does not know its values for goods A and B. The auctioneer asks Bidder 3 to report its marginal value for both goods A and B relative to its current demand, and the value function of Bidder 3 is fully revealed. There are three observations that should be made at this point. First, the standard Walrasian aggregate demand at $p(5)$ is $(0, 1)$, already below the supply. The excess demand for the main economy is $Z(N, 5) = (0, 0)$ indicating that the main economy is cleared. However, the cumulative excess demand that accounts for the main economy and all marginal economies is still positive since the marginal economy for Bidder 1 is not yet cleared and the auction must continue.

In the last stage, clock prices rise from $p(5) = (3, 5)$ to $p(7) = (3, 7)$. At $p(7)$, the marginal economy $E(N_{-1})$ is finally cleared, and the cumulative excess demand turns zero. Bidder 1 is awarded good A and charged $y_1^V = \hat{v}_3(AB) - \hat{v}_2(B) = 1$. Bidder 2 gets good B and pays $y_2^V = \hat{v}_3(AB) - \hat{v}_1(A) = 5$.

5 Implementation Issues

In this section, we discuss several issues related to the implementation of the iterative ascending auction from Section 4. Topics include providing bidders with feedback during the auction, the problem of indifferent bidders and reducing the number of elicitation queries.

Another implementation issue is an adaptation of our auction design to discrete prices. It is of general interest since most dynamic auctions in practice use discrete prices (with large increments) for elicitation. The complete specification of an iterative ascending auction for a discrete non-decreasing price path can be found in Appendix A.1.

5.1 Providing Feedback

One of the most important advantages of dynamic auctions is the ability to inform bidders about their prospective winnings/payments and the current level of competition in the midst of the auction process. For our auction, the relevant feedback can be as follows.

For winnings and payments, the auctioneer can inform bidder i about its current tentative assignment $x_i^*(N,t)$ and its current tentative payment calculated using (4.15). However, such feedback might be frustrating for bidder i when $x_i^*(N,t) \notin \widehat{\Delta}_i(t)$. A more sophisticated approach is to inform bidder i about its winning allocation and payment in a constrained winner determination problem in which bidder i is restricted to win a bundle from $\widehat{\Delta}_i(t)$.[19]

For the current level of competition, dynamic auctions traditionally disclose some measure of excess demand. For our design, excess demand $Z(t)$ defined in (4.13) is suitable for this purpose as it matches the closing condition and the rule for incrementing clock prices (note that $Z(t)$ is already partially revealed to bidders via price increments).

When providing bidders with feedback, the auctioneer should proceed with caution. Detailed feedback, such as reporting excess demand for the main economy $z(N,t)$ or providing bidders with their likely winnings and payments, can sometimes leak information about the cleared/uncleared status of the main economy and create incentive problems. We discuss this issue in the next section.

[19] A constrained winner determination problem is (3.3) with an additional constraint requiring $x_i \in \widehat{\Delta}_i(t)$. The corresponding Vickrey payment for bidder i is calculated as $\sum_{j\in N_{-i}} \left[\hat{v}_j(x_j^*(N_{-i},t)) - \hat{v}_j(\tilde{x}_j(N,t))\right]$ where $\tilde{x}(N,t) = (\tilde{x}_1(N,t),...,\tilde{x}_n(N,t))$ solves the constrained optimization problem.

5.2 Indifferent Bidders

To implement the Vickrey outcome, the auction must clear the main economy and all marginal economies. Due to stronger competition, it is natural for the main economy to be the last one to clear, but it is possible that it clears prematurely before the closing condition is trigged (as in the example in Section 4.4). Once the main economy is cleared and the winning allocation is determined, bidders become indifferent among all feasible bids since their Vickrey payments do not depend on their own bids. Therefore, if bidders are somehow aware that the main economy is cleared, incentives for truthful bidding might be compromised. An indifferent bidder might decide to completely cease its bidding (a problem known as "quiet bidding") or, alternatively, bid in a way that inflates payments of other bidders (a problem known as "predatory bidding" or "spiteful bidding").

The main trigger for the problem of indifferent bidders is leaking information about the status of the main economy. The Combinatorial Clock Auction (CCA) is a good example of an auction design that can prematurely disclose such information. Levin and Skrzypacz (2016) show that the problem of indifference can lead to the existence of inefficient equilibria in CCA. The same critique would apply to our iterative ascending auction if bidders are informed about the status of the main economy. To avoid premature disclosure, the auctioneer can use a "cumulative version" of excess demand (like the one defined by (4.13)) for determining clock price increments to hide the status of the main economy until the end. Another partial remedy for this problem is replacing formula (4.13) with an alternative that targets marginal economies first and switches to targeting the main economy only after all marginal economies have been cleared.

Mishra and Parkes (2007) showed that restricting the preferences domain solves the indifferent bidders problem for their auction design. In particular, the *buyers are substitutes* (BAS) condition guarantees that marginal economies always clear before the main economy in their auction; and the BAS condition is satisfied when the preferences domain is limited to gross substitutes.

Unfortunately, this result does not extend to our auction due to a difference between two elicitation processes. Mishra and Parkes (2007) use a semi-truthful elicitation process where $\hat{v}_i(z,t) = v_i(z) - \alpha_i(t)$ for all $z \in \widehat{\Delta}_i(t)$ and $\hat{v}_i(z,t) = 0$ for all $z \notin \widehat{\Delta}_i(t)$. In contrast, formula (4.9) imputes a more aggressive estimate $p(t)\, z - gv_i(t,z) \geq 0$ for bundles $z \notin \widehat{\Delta}_i(t)$ to compensate for linear prices. As a result, the BAS condition no longer guarantees the natural clearing order.

To illustrate the difference, consider an example with three items (one unit of good A and two units of good B) and three bidders who have values for at most two units. Bidder 1 values good A at 2 and has no value for good B, while Bidder 2 has no value for good A and values each unit of good B at 5. Bidder 3 has values $v_3(A) = 7$, $v_3(B) = 6$, $v_3(AB) = 10$ and $v_3(BB) = 9$. Note that all these value functions satisfy the gross substitutes.

The partial progress of the ascending auction for this example is provided in Table 2. The main economy clears at $p(3) = (3, 3)$ when approximations of the value functions are $\hat{v}_1 = (2, 0, 2, 0)$, $\hat{v}_2 = (3, 3, 6, 6)$ and $\hat{v}_3 = (3, 3, 6, 6)$. Note that both $\hat{v}_2$ and $\hat{v}_3$ are not semi-truthful, and the payment calculation for Bidder 2 based on these approximations is 5 while her true Vickrey payment is only 4. The auction has to continue since economy $E(N_{-2})$ has not yet been cleared.

Intuitively, the elicitation process must uncover the efficient allocation in the marginal economy $E(N_{-2})$ which includes Bidder 3 winning both units of good B. To elicit Bidder 3's value for this bundle, the auction should increment the clock price for good A at a faster rate than the one for good B. But the auctioneer has no apparent way of knowing this since the main economy and the marginal economy $E(N_{-1})$ have excess demand for good B while only economy $E(N_{-2})$ has excess demand for good A.

Table 2: An Illustrative Example with Substitutes

	Bidder 1	*Bidder 2*	*Bidder 3*
Values for (A, B, AB, BB):	$(2, 0, 2, 0)$	$(0, 5, 5, 10)$	$(7, 6, 10, 9)$
Efficient Allocation:	$\emptyset$	BB	A
Vickrey Payments:	0	4	2

Clock Prices	$x_1(t)$	$x_2(t)$	$x_3(t)$	$z(N_{-1}, t)$	$z(N_{-2}, t)$	$z(N_{-3}, t)$	$z(N, t)$	$Z(t)$
$t: 0 \to 2$ $p(t) = (t, t)$	A	BB	AB	(0, 1)	(1, 0)	(0, 0)	(1, 1)	(1, 1)
$t: 2 \to 3$ $p(t) = (t, t)$	$\emptyset$	BB	AB	(0, 1)	(1, 0)	(0, 0)	(0, 1)	(1, 1)
$p(3) = (3, 3)$	$\emptyset$	BB	A	(0, 0)	(1, 0)	(0, 0)	(0, 0)	(1, 0)

Another trigger for the problem of indifferent bidders is specific to our auction design where bidders communicate their preferences via two classes of bids: demand bids and marginal value bids.[20] When it is very likely that winning bids will come from the demand bids, bidders might be indifferent when placing their marginal value bids. Surprisingly, the gross substitutes preference domain is subject to this critique due to the guaranteed existence of linear market clearing prices. With clock prices being partially driven by the excess demand in the main economy, there is a nontrivial chance that the price path will pass through

[20] I thank the anonymous referee for pointing out this interesting feature of the design.

the market clearing prices in which case winning bids necessarily come from the demand bids.

5.3 Reducing Elicitation Burden

Another important implementation concern for dynamic auctions is the number of queries needed to elicit sufficient amount of information to implement the desired outcome. For general settings, ascending price auctions proposed by Mishra and Parkes (2007) provide a comparative benchmark for testing the auction design proposed in Section 4. The comparative analysis can be carried out using simulations which would need to consider multiple variations of auction designs and an array of different value environments.

Without conducting simulations, it is still possible to identify environments in which our auction design performs well. For example, in a homogeneous good environment and bidders with decreasing marginal values, our ascending auction simplifies to the ascending auction with "clinching" developed by Ausubel (2004) and delivers all advantages associated with using linear prices for elicitation. For another example, consider an environment with two goods, A and B, available in multiple quantities and bidders who view these goods as close substitutes but also put some premium on symmetric bundles (i.e., bundles where the total quantity is more equally distributed between both goods).[21] In our design, the elicitation process reveals mostly symmetric bundles and avoids asymmetric bundles, such as the ones containing only A goods or only B goods. In contrast, the elicitation process in Mishra and Parkes (2007) uncovers the relative values of bundles strictly in the descending order of their value, and the revelation of valuable but asymmetric bundles cannot be avoided.[22]

In general, the elicitation process from Section 4 can require bidders to report their marginal values for a large number of bundles at a time. In practice, the auctioneer should invite bidders to submit marginal values only for bundles of their interest and use zero incremental values for bundles that do not receive bids. In addition, the auctioneer can avoid unnecessary elicitation by eliminating bundles that can be proven irrelevant by the time their values have to be reported. A bundle is irrelevant for a given bidder if it can be shown that assigning it to the bidder leads to inefficient allocations in all relevant economies. The inefficiency of allocation x for economy $E(M)$ can be proved as follows:

[21] This setting appears to fit with the Ireland Multi-Band Spectrum Auction held in 2012 where bidders valued symmetric holdings in 800 MHz and 900 MHz bands.

[22] In order to infer the relative value of its 10^{th} most preferred bundle, the bidder would have to reveal relative values for its 1^{st}-9^{th} most preferred bundles.

Proposition 6. *A feasible allocation* $x = (x_1, ..., x_n)$ *is inefficient for economy* $E(M)$ *if there exists a feasible allocation* $y = (y_1, ..., y_n)$ *and time* t *such that*

$$y_i \in \Delta_i(t) \quad \forall i \in M \qquad \text{and} \qquad \sum_M \hat{v}_j(x_j, t) < \sum_M \hat{v}_j(y_j, t). \tag{5.1}$$

Proposition 6 provides a direct way to test bundles for irrelevance during the auction by replacing $\Delta_i(t)$ with $\widehat{\Delta}_i(t)$. The full description of a test for irrelevance is provided in Appendix A.2. Providing bidders with irrelevant bundles should be done with care since it can potentially leak information about bids of other bidders.

6 Conclusion

The virtues of the VCG mechanism create a strong interest in dynamic auctions that can implement the Vickrey outcome. For limited settings, practical designs that implement this outcome are well known. For the general private-value setting, Mishra and Parkes (2007) developed an ascending price Vickrey auction that relies on non-linear and non-anonymous clock prices — a complication that is often difficult to implement in practice.

In this paper, we demonstrate that flexible elicitation tools can avoid complex prices and facilitate a simpler auction process. The key idea is using a combination of demand queries and marginal value queries to ensure that all required information is properly captured. We design an efficient iterative ascending auction based on the elicitation process that uses linear and anonymous prices. To the best of our knowledge, it is the first iterative auction that uses simple and practical elicitation methods to implement the Vickrey outcome in the general private-value setting.

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Declaration of Interest

I am an academic associate of Power Auctions LLC, a consultancy that provides auction design and software implementation services to governments, non-governmental organizations and commercial enterprises, and I have received fees from Power Auctions exceeding US$10,000 in the past three years. Power Auctions may be considered an "interested" or a "relevant" party to the research reported in this paper.

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A Appendix – Discrete Prices and Irrelevant Bundles

A.1 Discrete Version of the Ascending Auction

In the iterative ascending auction described in Section 4, we used an intra-round bidding approach to accommodate a continuous price path. However, most dynamic auctions used in applications utilize discrete prices, generally with large increments. Here we provide a description of our auction that works with any discrete non-decreasing price path.

In Section 4, the continuity of the price path is used to ensure that bidder i, when switching its demand at time t from $x_i(t-0)$ to $x_i(t)$, is indifferent between the old and new demands given prices $p(t)$, i.e.,

$$v_i(x_i(t-0)) \;-\; p(t)\,x_i(t-0) \;=\; v_i(x_i(t)) \;-\; p(t)\,x_i(t). \tag{A.1}$$

With discrete prices, this property is lost. However, the auctioneer can collect an extra piece of information from a switching bidder to restore its equivalent.

Suppose that the auctioneer announces a price vector $p(t)$ and bidder i replies with its demand $x_i(t)$ at each time $t \in \{0, 1, 2, ...\}$. In addition, at each time $t \geq 1$, the auctioneer asks bidder i to report the smallest discount on the price of bundle $x_i(t-1)$, denoted $d_i(t)$, that bidder i commands to demand bundle $x_i(t-1)$ at $p(t)$ instead of its current demand $x_i(t)$.[23]

The discrete version of the iterative ascending auction can be constructed as follows. The procedure for constructing the set of revealed bundles $\widehat{\Delta}_i(t)$ does not change, and all formulas that are not mentioned below carry over from Section 4. The modified definition of truthful bidding is as follows. Bidder i is said to *bid truthfully according to its value function* $v_i(.)$ at time $t \in \{0, 1, 2, ...\}$ if:

(a) Bidder i truthfully reports its demand-discount pair $(x_i(t), d_i(t))$ given $p(t)$:

$$\begin{aligned} x_i(t) &\in \arg\max_{z \in \Omega} \big[v_i(z) \;-\; p(t)\,z\big]; \\ d_i(t) &= \big[v_i(x_i(t)) \;-\; p(t)\,x_i(t)\big] \;-\; \big[v_i(x_i(t-1)) \;-\; p(t)\,x_i(t-1)\big]. \end{aligned} \tag{A.2}$$

(b) (same as in Section 4)

For bidder i, denote $dx_i(t) = x_i(t) - x_i(t-1)$ a change in demand at time t, and $dv_i(t) = p(t)\,dx_i(t) + d_i(t)$ an implied change in value at time t. Note that $dv_i(t) = v_i(x_i(t)) - v_i(x_i(t-1))$ when bidder i bids truthfully.

[23] Note that in practice, the auctioneer needs to ask for $d_i(t)$ only when bidder i demands bundle $z \notin \widehat{\Delta}_i(t-1)$. For $z \in \widehat{\Delta}_i(t-1)$, activity rule AR1 would limit $d_i(t)$ to a single number.

The discrete analog of the GARP condition is stated as:

$$(GARP) \qquad p(s)[x_i(s) \,-\, x_i(s')] \;+\; \sum_{u\,=\,s'}^{s+1} dv_i(u) \;\leq\; 0 \qquad \forall\; s \;<\; s' \in \{0,...,t\},$$

and the discrete analog of the GARP violation is given by:

$$gv_i(t,z) \;=\; \max_{s\in\{0,...,t\}} \left\{ p(s)[x_i(s) - z] \;+\; \sum_{u\,=\,t}^{s+1} dv_i(u) \;+\; p(t)[z - x_i(t)] \right\}. \quad \text{(A.3)}$$

The modified activity rules are as follows:

AR1: At any time t, a bundle-discount pair $(z,d) \in \Omega \times \mathbb{R}$ is unacceptable if either $gv_i(t,z) > 0$ or $z \geq x_i(s)$ for some time $s < t$ (when validating AR1, $gv_i(t,z)$ is calculated using (A.3) by setting $x_i(t) := z$ and $d_i(t) := d$).

AR2: (same as in Section 4)

The discrete analog of formula (4.8) is given by:

$$mv_i(z,t) \;=\; \begin{cases} -\sum\limits_{u\,=\,t}^{t'+1} dv_i(u) & \text{if } \exists\, t' \in \{0,...,t\} :\; x_i(t') = z \\ -\sum\limits_{u\,=\,t}^{t(z)+1} dv_i(u) \;+\; \widetilde{mv}_i(z) & \text{otherwise} \end{cases}. \quad \text{(A.4)}$$

Finally, the price adjustment process is modified as follows:

$$p^k(t) \;=\; \begin{cases} p^k(t-1)\,[1 \;+\; \epsilon] & \text{if} \quad Z^k(t-1) > 0 \\ p^k(t-1) & \text{if} \quad Z^k(t-1) = 0 \end{cases}. \qquad \text{(A.5)}$$

A.2 Test for Irrelevant Bundles

For economy $E(M)$ and bidder $i \in M$, the following steps have to be carried out to test bundle z for irrelevance at time t:

1. Find the maximal value that can be obtained by giving each bidder in M a bundle from its set of revealed bundles $\widehat{\Delta}_i(t)$ by solving the following problem:

$$H1 \;=\; \max_{y\,\in\,X} \;\textstyle\sum_{j\in M} \hat{v}_j(y_j,t) \qquad \text{s.t.} \quad y_j \in \widehat{\Delta}_j(t) \qquad \forall j \in M \qquad \text{(A.6)}$$

If solution does not exist, the test cannot proceed to step 2.

2. Find the maximal value that can be obtained by allocating bundle z to bidder i by solving the following problem:

$$H2 \;=\; \max_{x \in X} \; \textstyle\sum_{j \in M/\{i\}} \hat{v}_j(x_j, t) \qquad \text{s.t.} \qquad x_i = z \tag{A.7}$$

If $p(t)\, z \;-\; gv_i(z, t) \;\le\; H1 \;-\; H2$, then bundle z is irrelevant for bidder i in economy $E(M)$.

B Appendix B: Proofs

PROOF OF THEOREM 1. Suppose that all bidders bid truthfully. Given the adjustment rule and an iterative elicitation process, the procedure would eventually force each bidder to fully reveal its $v(.)$ at which point $\widehat{\Delta}_i(t) = \Omega$ for all $i \in N$ and the closing condition is satisfied. Therefore, the procedure has to end in finite time T. For a straightforward elicitation process, $x^*(N, T)$ is efficient by Proposition 1 and $p_i^V = y_i^V$ by Proposition 2. Now suppose that bidder i deviates from truthful bidding. First, note that bidder i cannot block the auction from closing. Second, denote $\tilde{v}_i(z) = \hat{v}_i(z, T)$ which is well-defined. Due to a straightforward elicitation process, $\widehat{\Delta}_j(T) \subseteq \Delta_j(T)$ for all bidders in N_{-i}, and $\widehat{\Delta}_i(T) \subseteq \tilde{\Delta}_i(T)$ where $\tilde{\Delta}_i(T)$ is the analog of $\Delta_i(T)$ for $\tilde{v}_i(.)$. By Propositions 1 and 2, the auction outcome corresponds to a Vickrey outcome for the value profile $(\tilde{v}_i, v_{-i})$ which is weakly dominated for bidder i. □

PROOF OF PROPOSITION 3. (Necessity) For AR1, suppose that bidder i bids truthfully according to $\tilde{v}_i(.)$ and z is her true demand at time t. Then $gv_i(t, z) = 0$ by the Afriat's theorem. If $z \ge x_i(s)$ for some time $s < t$, then it must be the case that z and $x_i(s)$ are both optimal demands at time t. For AR2, by revealed preference,

$$\begin{aligned}
\tilde{v}_i(z) \;&\le \min_{s \in [0,t]} \{\tilde{v}_i(x_i(s)) + p(s)[z - x_i(s)]\} \\
&= \min_{s \in [0,t]} \left\{ \tilde{v}_i(x_i(t)) - \textstyle\int_s^t p(u) dx_i(u) + p(s)[z - x_i(s)] \right\} \\
&= \tilde{v}_i(x_i(t)) + p(t)[z - x_i(t)] - gv_i(t, z)
\end{aligned}$$

and the marginal value between bundles z and $x_i(t)$ is given by $\tilde{v}_i(z) - \tilde{v}_i(x_i(t)) \le p(t)[z - x_i(t)] - gv_i(t, z) = \widetilde{mv}_i(z)$. Then AR2 never overconstrains a bidder who bids according to some value function $\tilde{v}_i(.)$. *(Sufficiency)* Suppose that both AR1 and AR2 hold on $[0, t]$. Construct $\tilde{v}_i(.)$ as

$$\tilde{v}_i(z) = \begin{cases} p(t)\, x_i(t) - \int_{t(z)}^t p(u) dx_i(u) + \widetilde{mv}_i(z) & z \in \widehat{\Delta}_i(t) \\ 0 & z \notin \widehat{\Delta}_i(t) \end{cases}$$

where $t(z) = s'$ and $\widetilde{mv}_i(z) = 0$ if bundle z was demanded by bidder i at time $s' \leq t$. Now we show that $\tilde{v}_i(.)$ rationalizes bidding of bidder i at any time $s \in [0,t]$. For bundle z that was demanded at some time $s' \in [0,t]$ (i.e., $x_i(s') = z$), the bidding on $[0,t]$ is rationalized since $\tilde{v}_i(z) - p(s)\,z \leq \tilde{v}_i(x_i(s)) - p(s)\,x_i(s)$ is equivalent to $p(s)[x_i(s) - z] + \int_s^{s'} p(u)dx_i(u) \leq 0$ which is satisfied due to AR1. For bundle $z \in \widehat{\Delta}_i(t)$ that was never demanded explicitly but was added to the set of revealed bundles at time $t(z)$ (so $z \geq x_i(t(z))$):

- For $s \leq t(z)$, the bidding on $[0,t]$ is rationalized since $\tilde{v}_i(z) - p(s)\,z \leq \tilde{v}_i(x_i(s)) - p(s)\,x_i(s)$ is equivalent to $\widetilde{mv}_i(z) \leq -p(s)[x_i(s) - z] - \int_s^{t(z)} p(u)dx_i(u) \leq p(t(z))[z - x_i(t(z))] - gv_i(t(z), z)$ which is satisfied due to AR2.

- For $s > t(z)$, we show that $gv_i(t(z), z) = gv_i(s, z)$ (i.e., the GARP violation for bundle z is not going to increase over time). Then the bidding is rationalized by the same argument as for the case $s \leq t(z)$. Suppose that there exist $s' \in (t(z), s]$ such that the GARP violation for bundle z is strictly higher for s' than for $t(z)$ (i.e., $p(s')[x_i(s') - z] + \int_{s'}^{s} p(u)dx_i(u) > p(t(z))[x_i(t(z)) - z] + \int_{t(z)}^{s} p(u)dx_i(u)$) But then $p(s')[x_i(s') - x_i(t(z))] + \int_{s'}^{t(z)} p(u)dx_i(u) > [p(s') - p(t(z))][z - x_i(t(z))] \geq 0$ where the last inequality implies violation of AR1 at time s'. Hence, $gv_i(t(z), z) = gv_i(s, z)$.

□

Lemma 1. *If bidder i bids truthfully according to $v_i(.)$ on $[0,t]$, then for all $z \in \widehat{\Delta}_i(t)$, the revealed marginal value of bundle z relative to bundle $x_i(t)$ is the true marginal value, i.e., $mv_i(z,t) = v_i(z) - v_i(x_i(t))$.*

Proof. First, suppose that there exists $t' \in [0,t)$ such that $x_i(t') = z$. Denote $t_1, ..., t_m$ all times in the interval $[t', t]$ when bidder i changed its demand. Given the continuous price path $p(.)$, bidder i who bids truthfully is indifferent at all switch points, i.e.,

$$\begin{aligned} v_i(z) - p(t_1)\,z &= v_i(x_i(t_1)) - p(t_1)\,x_i(t_1) \\ ... &\;...\; ... \\ v_i(x_i(t_{m-1})) - p(t_m)\,x_i(t_{m-1}) &= v_i(x_i(t)) - p(t_m)\,x_i(t) \end{aligned}$$

Then by (4.8) $v_i(z) = v_i(x_i(t)) - \int_{t'}^{t} p(u)dx_i(u) = v_i(x_i(t)) + mv_i(z,t)$. Second, if $z \in \widehat{\Delta}_i(t)$, but it was never explicitly demanded by bidder i, then by (4.8) and the definition of truthful bidding, we have $mv_i(z,t) = \int_t^{t(z)} p(u)dx_i(u) + \widetilde{mv}_i(z) = v_i(z) - v_i(x_i(t))$. □

PROOF OF PROPOSITION 4. Formulas in (4.1) and (4.9) indeed represent an elicitation process since they uniquely define $\widehat{\Delta}_i(t)$ and $\hat{v}_i(.,t)$ at each time t and each bidder i. By construction, the elicitation process is iterative and ascending since $p(.)$ is nondecreasing. *(Straightforward)* Bidder i bids truthfully according to $v(.)$ on $[0, t]$. For any bundle $z \in \widehat{\Delta}_i(t)$, $\delta_i(z, t) = \delta_i(x_i(t), t)$ by Lemma 1. For any bundle $z \notin \widehat{\Delta}_i(t)$, $\delta_i(z, t) \leq \delta_i(x_i(t), t)$ by (4.6). Then if $z \in \widehat{\Delta}_i(t)$, then $z \in \Delta_i(t)$. *(Monotonic)* Consider time $t' < t$ and bundle $z \notin \widehat{\Delta}_i(t)$. Then $\hat{v}_i(z, t) - \hat{v}_i(z, t') = [p(t) - p(t')]\, z - [gv_i(t, z) - gv_i(t', z)] \leq \hat{v}_i(x_i(t'), t) - \hat{v}_i(x_i(t'), t')$. But then $\hat{v}_i(z, t) - \hat{v}_i(z, t') \leq \hat{v}_i(y, t) - \hat{v}_i(y, t')$ for any $y \in \widehat{\Delta}_i(t')$. $\square$

PROOF OF PROPOSITION 5. If economy $E(M)$ is cleared at time t, then $0 \in Z(M, t)$ by the definition of $Z(M, t)$. For the converse, suppose that $0 \in Z(M, t)$, but $E(M)$ is not cleared at t. Then for any tentative assignment $x^*(M, t)$, there is at least one bidder $i \in M$ such that $x_i^* \notin \widehat{\Delta}_i(t)$. For such bidder i, there is at least one good for which demand $d \in D_i(t)$ strictly exceeds the tentative award x_i^* since otherwise, $x_i^* \geq d$ implying that $x_i^* \in \widehat{\Delta}_i(t)$. But then $0 \notin Z(M, t)$. $\square$

PROOF OF THEOREM 2. By Proposition 4, the elicitation process is straightforward and iterative. For each economy $E(M)$ that is not cleared at integer time t, $Z^k(t) > 0$ if good k is overdemanded in at least one relevant economy, and its price will be increased by $\epsilon\, p^k(t) > 0$ on the time interval $(t, t+1]$. Increasing clock prices for overdemanded goods cause an increase in $\hat{v}_i(z, t)$ for all bundles $z \in \widehat{\Delta}_i(t)$ for bidder i who demands overdemanded item at time t (satisfies the adjustment rule (3.9) on a sufficiently long time interval). Then the Vickrey outcome is implemented by Theorem 1. $\square$

PROOF OF PROPOSITION 6. A feasible allocation x is inefficient since:

$$\sum_{j \in M} v_j(x_j) \;=\; \sum_{j \in M} [\hat{v}_j(x_j, t) + \delta_j(x_j, t)] \;<\; \sum_{j \in M} [\hat{v}_j(y_j, t) + \delta_j(y_j, t)] \;=\; \sum_{j \in M} v_j(y_j)$$

$\square$