Exposure vs. Free-Riding in Auctions with Incomplete Information

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November 2010

Abstract

The recent auctions literature has devoted much attention to mechanisms that allow package bidding: all-or-nothing bids for sets of items. Introducing package bids can improve efficiency by reducing the large bidder's "exposure" risk of winning undesirable combinations of items. However, package bids can also create a free-rider problem for small bidders since they need to compete jointly against large package bidders, potentially reducing efficiency. The inherent asymmetry among different package bids complicates an equilibrium treatment of the first-price package auction, the most widely used package format empirically, preventing a full analysis of the costs and benefits of allowing package bids.

This paper makes progress in solving for Bayesian-Nash equilibria of the first-price package auction. We develop a new computational method which is based on a complementarity formulation of the system of equilibrium inequalities. Additionally, we establish existence of equilibrium for special cases. Our analysis shows that introducing package bidding can significantly improve efficiency when the exposure risk faced by bidders is large, but it can reduce efficiency otherwise. We also compare the first-price package auction with other leading package alternatives, including core-selecting auctions. Surprisingly, in the environment considered, the first-price package auction performs reasonably well, with respect to both revenue and efficiency, despite the presence of a strong free-rider problem.

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1 Introduction

The recent auction literature has devoted a lot of attention to auction mechanisms that can be used to allocate multiple items simultaneously. Such strong interest is not surprising since the majority of recent high-stake auctions involved selling many items at once to participants with non-trivial interests in different combinations of these items. For example, in spectrum auctions, governments sell licenses to use the airwave spectrum in many geographical regions to telecommunication companies whose business plans only match some particular subsets of licenses. Other examples include treasury auctions, bankruptcy auctions and numerous private and public procurement auctions.

Bidders in these auctions can often have very specific preferences for some bundles of items offered in the auction. For example, a buyer's value for a pair of objects can be higher than the standalone values of the individual components because the combination of items generates additional synergetic value. In other words, a bidder may value some objects as complements. In such environments, standard independent single-item auctions might fail to produce an efficient allocation because they do not allow bidders to express their possible synergies across multiple items.

For the ease of exposition, consider a simple illustrative example with just two items, "East" and "West" (which might represent spectrum licenses for the Eastern and Western parts of a country), offered for sale and three bidders. The "global" bidder regards both East and West as perfect complements, receiving positive value from the package {East, West} but obtains no value from either item when acquiring them individually. The other bidders, usually referred to as "local" bidders, obtain value only from one item. Local Bidder 1 is interested only in East and obtains no value for West while the local bidder 2 obtains value only for West and has no interest in East. In case the auctioneer decides to sell the items by means of two independent auctions, the global bidder is exposed to a positive chance of being forced to buy just East or just West when she wins only in one auction but loses in the other one. Since the global bidder views both items as perfect complements, acquiring just one of them is totally worthless for her.

This phenomenon, known as the *exposure problem*, can negatively affect efficiency of the auctions because exposed bidders will strategically underbid or overbid in an attempt to avoid paying positive prices for good-for-nothing packages.

The auctioneer can easily accommodate bidders with synergies and avoid the exposure problem altogether by simply auctioning all items simultaneously and allowing bidders to submit package bids, i.e., all-or-nothing bids for subsets of items specified by bidders themselves. In the example from the previous paragraph, the global bidder will be able to submit a package bid for the actually desired combination of items, i.e., for both East and West bundled together. Such package bid completely eliminates any exposure risk because the global bidder can win either both items or no items at all with zero probability of getting just one of them.

However, while package bids solve all exposure concerns, they also can introduce a *free-rider problem*, sometimes referred to as the threshold problem in the auction literature. The issue can be easily demonstrated using the illustrative example introduced above. Consider a first-price auction where the global bidder submits a package bid B for the bundle of East and West while local bidders bid b_1 and b_2 on the corresponding items. The local bidders win their items whenever their total bid is higher than the package bid of the global bidder ($b_1 + b_2 > B$). Without an ability to explicitly coordinate their actions, both local bidders have incentives to free ride on each other by reducing their individual bids. Observe that it is possible for a local bidder to get the desired item for free when the other local bidder outbids the package bid of the global bidder alone[1]. Such free-riding motives can easily mitigate any possible efficiency and revenue gains achieved by the package bidding and further degrade the performance of the package design.

Therefore, the overall effect of the package bidding on the auction performance characteristics is ambiguous. On the one hand, package bids might increase efficiency and revenue by eliminating the exposure problem for bidders who view objects as complements. On the other hand, the presence of package bids might negatively affect efficiency and revenue when some bidders without a fundamental conflict of interest have incentives to free ride on other bidders' bids.

The most widely used package design in applications is the sealed-bid first-

[1]Consider the following bid data: $B = 5,\ b_1 = 0,\ b_2 = 6$. The local bidder 1 wins her item and pays zero since $b_1 + b_2 = 6 > 5 = B$.

price package auction. Virtually all procurement auctions which allow bidders to submit at least some package bids use the pay-as-bid pricing rule. Examples include auctions of bus routes in London,[2] auctions for milk providers in Chile[3] and IBM-Mars procurement auctions.[4] Such features as resistance to collusion, participation encouragement and transparency (winners pay what they bid) explain the popularity of the sealed-bid pay-as-bid auctions.[5] Therefore, the first-price auction is the most relevant auction format to study the consequences of package bidding from a policy-making perspective. However, an inherent structural asymmetry between bidders' values and bids is crucial for having non-trivial versions of both the exposure and free-rider problems. This requirement is unfortunate since a general theoretical analysis of first-price auctions in asymmetric environments has proved to be very tedious.

In this paper, we consider a simple stylized model with two items in which one bidder values them as perfect complements. First, we perform a Bayesian-Nash equilibrium analysis for both the package and non-package versions of the first-price auction, including existence proofs for some instances of the model. Second, we develop a novel numerical technique that is used to approximate equilibrium bidding functions in both auctions. Finally, we investigate the impact of package bidding on the performance of the first-price auctions and compare the first-price package auction with other leading package alternatives.

Results on existence of the monotone Bayesian-Nash equilibrium in the first-price package auctions are related to the large body of literature on existence of the equilibrium in first-price auctions for a single item (Maskin and Riley (2000), Lebrun(1999))[6] and in games of incomplete information in general (Radner and Rosenthal (1982), Milgrom and Weber (1985), Athey (2001)). The numerical technique developed here in order to approximate equilibrium strategies is related to the literature on numerical methods designed to solve asymmetric first-price auctions (Marshall et al (1994), Riley and Li (1997), Bajari (2001)) and general games of incomplete information (Armantier and Richard (1997)).

[2]See Cantillon and Pesendorfer (2007).
[3]See Epstein, Henríques, Catalán, Weintraub and Martínez (2002).
[4]See Hohner, Rich, Ng, Reed, Davenport, Kalagnanam, Leen and An (2002).
[5]See Cramton (1998) for discussion.
[6]See also Lebrun(1996), Bajari(1997), Lizzeri and Persico (2000).

This paper is a contribution to the literature on package auctions. The best-known package auction is the Vickrey-Clarke-Groves (VCG) mechanism, which was introduced in the classic theory of auctions and public choice. Vickrey (1961) looked into multiple-unit auctions for homogeneous goods while Clarke (1973) and Groves (1973) studied public choice models in potentially heterogeneous environments.

Other auction formats with package bids have been extensively studied in complete information settings. Bernheim and Whinston (1986) developed a general theory of the first-price package auctions. Day and Milgrom (2008) defined a class of alternative package designs which came to be known as *core-selecting auctions*. A particular interesting subclass of core-selecting auctions are auctions that minimize seller's revenue subject to the core constraints. Day and Milgrom (2008) showed that such *minimum-revenue core auctions* maximize bidders' incentives for truthful reporting[7]. There are a lot of examples of minimum-revenue core auctions in the auction literature including "the ascending proxy auction" introduced by Ausubel and Milgrom (2002, 2006)[8] and the nearest-Vickrey package auction developed by Day and Raghavan (2007) and Day and Cramton (2009).

Some recent papers have begun to explore the comparison among core-selecting auctions in an incomplete information environment using the simple model similar to the one used in this paper. Erdil and Klemperer (2009) introduced another subclass of minimum-revenue core payment rules, which are referred to as "reference rules," and argued that such payment rules perform better than any other minimum-revenue core rules because they minimize bidders' marginal incentives to deviate. Sano (2010) analyzed the "ascending proxy auction" in a simple setup with independent values. Goeree and Lien (2009) showed that in general environments with complementarities a core-selecting auction that shares the dominant strategy property of the VCG does not exist. Ausubel and Baranov (2010) compared a variety of minimum-revenue core payment rules and showed that a positive correlation among bidders' values can have a dramatic impact on the magnitude of the similar free-rider problem which also plagues all core-selecting auctions in

[7]Note that the first-price package auction belongs to the class of core-selecting auctions, but in this case the seller's revenue is maximized subject to the core constraints, unlike in "second-price-like" minimum-revenue core auctions.

[8]A closely related auction procedure was developed independently by Parkes and Ungar (2000) and Parkes (2001).

environments with complementarities.

Chernomaz and Levin (2010) studied the effects of package bidding on the sealed-bid first-price auctions in the experimental setting. They considered a similar model with the global bidder, who has positive synergies, and two local bidders. Regretfully, they made a simplifying assumption about the perfect correlation between local bidders' values. While this assumption significantly simplifies their theoretical analysis, it also effectively eliminates the exposure problem[9] from their consideration. As has been noted above, the exposure worries arising from synergies, not synergies without exposure risks, are the leading motivator for using package designs. For that reason, our model explicitly allows exposure outcomes to be part of the equilibrium.

The model adapted here is a variation of the original model introduced by Krishna and Rosenthal (1996) to study the exposure problem in second-price auctions with synergies. The main model of the paper is an incomplete-information version of the illustrative example model used throughout this introduction. The global bidder obtains value u from winning both items but gets zero value from getting only one of them. Each local bidder i values her corresponding item at v_i and obtains no value from the other item. As in Ausubel and Baranov (2010), the local bidders' values are perfectly correlated with probability γ and independently distributed with probability $1-\gamma$. Moreover, local bidders only get to observe their values, but they are unaware of whether their values are perfectly correlated ($v_1 = v_2$) or independent. Thus, γ parameterizes a family of distributions that permits the correlation between local bidders' values to be varied continuously from zero to one.

There are several reasons for considering a class of models with positive correlation among bidders' values. First, in important applications such as spectrum auctions, the correlations among bidders' values can be significant since similar bidders are likely to use spectrum licenses to deploy the same telecommunication technology. Second, Ausubel and Baranov (2010) found that positive correlation can considerably affect the performance of different package-bidding designs. Finally, when package bids are not allowed, the positive correlation can also mitigate the exposure problem. For instance, in

[9] The exposure problem still exists in the experiment part of their paper. However, the lack of the exposure problem in the underlying theoretical model provides an alternative explanation for their main results.

case the values of local bidders are perfectly correlated ($\gamma = 1$) and both local bidders follow the same bidding strategy[10], the global bidder can completely avoid any exposure risk by submitting the same bid in both auctions even though her preferences exhibit the extreme form of complementarities.

The model clearly demonstrates the mechanics of the trade-off between the exposure and free-rider problems. When the exposure concerns are weak, package bids can significantly hurt efficiency, but they can sharply improve it when the exposure risk is relatively high. Further, we show that package bids can substantially improve efficiency, even when there is no exposure problem at all, simply because the bid asymmetries introduced by package bids might compensate for the prior distributional asymmetries. Surprisingly, the first-price package auction is more efficient when the global bidder has a distributional advantage over the local bidders. Finally, several examples demonstrate that the package auction can also generate higher revenues in more competitive environments.

The paper proceeds as follows. The model is described in Section 2. Section 3 contains results of the Bayesian-Nash equilibrium analysis of the first-price auction with and without package bids. The numerical approach which is used to approximate equilibrium bidding functions is described in Section 4. Several examples that compare the relative performance of the package and non-package versions of the first-price auction are presented in Section 5. We compare the first-price package auction with the other leading package alternatives in Section 6. Section 7 concludes. Most proofs are relegated to the Appendix.

2 Model

The model used here closely follows the model developed in Ausubel and Baranov (2010). It consists of two items offered for sale, two local bidders and one global bidder. Local bidders, denoted 1 and 2, are interested only in one item and receive no extra utility from acquiring the second item. Their private values are denoted v_1 and v_2, respectively. The global bidder wants to acquire two items and gets zero utility from owning just one item. The

[10] The solution concept is Bayesian-Nash equilibrium. Since the joint distribution of values will be symmetric with respect to the two local bidders, and we will limit attention to Bayesian-Nash equilibria that are symmetric with respect to the two local bidders.

value she receives in case she gets both items is denoted u. All bidders are risk-neutral with quasilinear preferences. Thus, the payoff of the local bidder i if she wins an item at price p_i, is $v_i - p_i$. The payoff of the global bidder, if she wins two items for a total price p, is $u - p$; while, if the global bidder wins only one item at price p, her payoff is simply $-p$.

With probability $\gamma \in [0, 1]$, both local bidders have exactly the same value v, which is drawn from the distribution on $[0, \bar{v}]$ defined by a cumulative distribution function $F(v)$ with atomless probability density function $f(v)$. With probability $1 - \gamma$, the values, v_1 and v_2, of the local bidders are drawn from the same distribution $F(v)$ independently from each other. The value of the global bidder, u, is independently drawn from the distribution on a $[0, \bar{u}]$ described by a cumulative distribution function $G(u)$ with atomless density $g(u)$. For the ease of exposition, we assume that $\bar{u} = 2\bar{v}$.

The assumption of independence between values of the global bidder and local bidders is reasonable since many bidder-specific characteristics such as cost structure and the scale of operations may be substantially different for the global bidder and local bidders. Meanwhile, both local bidders are alike in a sense of demanding only one item, so it is likely that their values are similar. For example, in a spectrum auction, the local bidders might be two firms that plan to put the same telecommunication technology in operation in two different geographic areas with similar demographic characteristics.

Parameter γ controls the amount of correlation between local bidders' values. For example, $\gamma = 0$ and $\gamma = 1$ correspond to the cases of independent values and perfectly correlated values respectively. The local's bidder value model can be summarized by the conditional cumulative distribution function of the local bidder i that defines her probability assessment of bidder's j valuations given her value v_i:

$$F_L(v_j | v_i = s) = \begin{cases} (1-\gamma)F(v_j) & 0 \leq v_j < s \\ (1-\gamma)F(v_j) + \gamma & s \leq v_j \leq \bar{v} \end{cases} \qquad i \neq j$$

Without loss of generality, our attention is limited to the first-price package auction where any bidder is allowed to submit only one bid. While impractical in general environments, this limitation has no implications for the analysis of the model because of the perfect complementarity nature of the bidders' preferences[11] in our model. For example, the global bidder values

[11] All bidders weakly prefer to bid for their desired bundles.

only a bundle of two items and her bid B is interpreted as a package bid for two items[12]. Each local bidder i is interested only in one item and her bid b_i expresses her willingness to pay b_i for the item.

The first-price package auction proceeds in the following manner. First, all bidders submit their bids to the auctioneer who then chooses an allocation which maximizes total welfare with respect to the bids. In this simple model, only two outcomes are possible. If the package bid of the global bidder is greater than the sum of the local bidders' bids ($B > b_1 + b_2$), the global bidder receives both items and pays B. The local bidders win the auction and receive one item each whenever the sum of their bids is higher than the package bid of the global bidder ($B < b_1 + b_2$). In this event, both local bidders are required to pay their respective bids. Ties are resolved using a fair randomizing device.

When package bids are not allowed, both items are auctioned simultaneously using two independent first-price auctions (or, equivalently, a pay-as-bid auction where bidders submit demand curves in case items are homogeneous). Naturally, the global bidder participates in both auctions by submitting two separate bids, b_1^g and b_2^g, each for the corresponding item. There are several possible outcomes. First, the global bidder can win both items when her bids are higher than the corresponding bids of the local bidders ($b_1^g > b_1$ and $b_2^g > b_2$) and pay $b_1^g + b_2^g$. Second, the global bidder can end up winning only one item, in which case she receives no value from the acquired item but pays the amount of her winning bid. Finally, the global bidder can lose in both auctions and pay nothing if both of her bids are smaller than that of local bidders ($b_1^g < b_1$ and $b_2^g < b_2$). A local bidder i wins the desired item when her bid b_i is higher than the corresponding bid of the global bidder b_i^g and pays b_i. Ties are resolved independently across auctions using a fair randomizing device.

The majority of proofs in Section 3 and the numerical technique discussed in Section 4 are based on the discrete bidding regime. In the discrete bidding regime all bids by the global bidder are constrained to a countable set of points $\Delta^G = B^0 < B^1 < \ldots < B^j < \ldots$ and bids by local bidders are restricted to a similar countable set of points $\Delta^L = b^0 < b^1 < \ldots < b^i < \ldots$

[12] Since the global bidder can only submit one bid, there are no inefficiencies arising from the strategic price discrimination in the sense of Cantillon and Pesendorfer (2007).

where $B^0 = b^0 = 0$.

In most applications, the bidding sets Δ^G and Δ^L are equally spaced bid grids characterized by a certain increment, like a dollar or a penny. However, it is possible that the global bidder, when bidding for a package of items, is restricted to bid in larger increments, say twice the minimum increment on any individual item. It is also assumed that Δ^G and Δ^L are unbounded.

Some proofs are based on a continuous bidding regime where $\Delta^G = [0, +\infty)$ and $\Delta^L = [0, +\infty)$.

We proceed with the equilibrium analysis of the first-price package auction.

3 Equilibrium Analysis

This section develops the equilibrium existence results for the first-price auction with and without package bids.

3.1 Equilibrium Analysis of the First-Price Package Auction

Since the bidding sets Δ^G and Δ^L are unbounded, sufficiently high bids from these sets will never be a part of any equilibrium of the first-price auction. Without loss of generality, the global bidder selects her bid from a finite set $S^G = \{B^0, B^1, ..., B^{k_G}\}$ and local bidders choose their actions from a finite set $S^l = \{b^0, b^1, ..., b^{k_l}\}$ where k_G and k_l are defined as follows:

$$k_G = \{j \in \mathbb{N} : B^j \leq \bar{u},\ B^{j+1} > \bar{u}\} \qquad k_l = \{i \in \mathbb{N} : b^i \leq \bar{v},\ b^{i+1} > \bar{v}\}$$

A pair of functions, $B(u) : [0, \bar{u}] \to S^G$ and $\beta(v) : [0, \bar{v}] \to S^l$, forms a pure symmetric Bayesian-Nash equilibrium if the following two conditions hold:

$$\begin{gathered} \forall v \in [0, \bar{v}] \quad \exists i \in \mathbb{N} : \pi_L(v, b^i) \geq \pi_L(v, b^k) \quad \forall k \in \mathbb{N} \\ (\text{i.e.} \quad \beta(v) = b^i) \end{gathered} \tag{3.1}$$

$$\begin{gathered} \forall u \in [0, \bar{u}] \quad \exists j \in \mathbb{N} : \pi_G(u, B^j) \geq \pi_G(u, B^k) \quad \forall k \in \mathbb{N} \\ (\text{i.e.} \quad B(u) = B^j) \end{gathered} \tag{3.2}$$

where $\pi_L(v, b^i)$ denotes the expected payoff of a local bidder with value v and bid b^i and $\pi_G(u, B^j)$ denotes the expected payoff of the global bidder with value u and a package bid B^j.

Conditions (3.1) and (3.2) are incentive compatibility (IC) constraints for local bidders and the global bidder respectively. Note that since $\pi_L(v, b^0) \geq 0$ for any $v \in [0, \bar{v}]$ and $\pi_G(u, B^0) \geq 0$ for any $u \in [0, \bar{u}]$ individual rationality (IR) constraints follow naturally from the IC constraints.

3.1.1 Independent Values ($\gamma = 0$)

When values of local bidders are independent, the expected probability of winning depends only on the bidder's bid since her value does not provide any inference about her opponents' values. The expected profits in this case are given by:

$$\begin{aligned} \pi_L(v, b^i) &= (v - b^i) Pr(\text{Locals win}|b^i) = (v - b^i) Pr_L^i \\ \pi_G(u, B^j) &= (u - B^j) Pr(\text{Global wins}|B^j) = (u - B^j) Pr_G^j \end{aligned} \tag{3.3}$$

For a fixed profile of the opponents' strategies, an increase in a bidder's bid never reduces her probability of winning. Therefore, the following inequalities hold:

$$\begin{aligned} Pr_L^i &\leq Pr_L^{i+1} \quad \forall i \in \mathbb{N} \\ Pr_G^j &\leq Pr_G^{j+1} \quad \forall j \in \mathbb{N} \end{aligned} \tag{3.4}$$

Lemma 1 establishes the monotonicity property for equilibrium bidding functions.

Lemma 1. *If local bidders' values are independent ($\gamma = 0$), equilibrium bidding functions $\beta(v)$ and $B(u)$ are nondecreasing.*

Proof. See Appendix. □

Note that Lemma 1 guarantees that sets $\mathbf{b}^i = \{v \in [0, \bar{v}] : \beta(v) = b^i\} \quad \forall i$ and $\mathbf{B}^j = \{u \in [0, \bar{u}] : B(u) = B^j\} \quad \forall j$ are convex.

Lemma 2 characterize a Bayesian-Nash equilibrium in pure monotone strategies.

Lemma 2. *If local bidders' values are independent ($\gamma = 0$), a pure-strategy Bayesian-Nash symmetric equilibrium is characterized by a pair of step-functions with the following functional forms:*

$$\beta(v) = \begin{cases} b^i & \text{if } v \in [s_i, s_{i+1}) \quad 0 \leq i \leq r \\ b^r & \text{if } v = s_{r+1} \end{cases} \tag{3.5}$$

and

$$B(u) = \begin{cases} B^j & \text{if } u \in [t_j, t_{j+1}) \quad 0 \leq j \leq q \\ B^q & \text{if } u = t_{q+1} \end{cases} \tag{3.6}$$

where

1. $0 = s_0 < s_1 \leq ... \leq s_{r+1} = \bar{v}$
   $0 = t_0 < t_1 \leq ... \leq t_{q+1} = \bar{u}$
2. $0 \leq r \leq \min[k_L, r^*(q)]$ *where* $r^*(q) = \{i \in \mathbb{N} : b^{i-1} \leq B^q,\ b^i > B^q\}$
   $0 \leq q \leq \min[k_G, q^*(r)]$ *where* $q^*(r) = \{j \in \mathbb{N} : B^{j-1} \leq 2b^r,\ B^j > 2b^r\}$

Proof. By Lemma 1, any equilibrium involves a pair of nondecreasing functions. Giving finite discrete strategy sets, the step function is the only possible functional form. Note that we have fixed the actions of all bidders at "jump" points. Technically, there are a lot of equilibria that assign other actions to "jump" points but otherwise they are equivalent to the one described in (3.5) and (3.6) since densities $f(v)$ and $g(u)$ are atomless. Indexes r and q are the highest bid levels played in the equilibrium with positive probability. Indexes $r^*(q)$ and $q^*(r)$ are defined such that the bidder with the highest type never bids above the bid that outbids the maximum possible bid from the opposing side. This is a standard conclusion for the first-price auctions. The fact that all probabilities of winning, Pr_L^i and Pr_G^j are strictly positive is reflected in strict inequalities: $s_0 < s_1$ and $t_0 < t_1$. □

Proposition 1 (Discrete Bidding)**.** *There exists a symmetric (across local bidders) Bayesian-Nash equilibrium of the first-price package auction in pure nondecreasing strategies when values of the local bidders are independent, i.e.,* $\gamma = 0$.

Proof. Existence of the equilibrium when local bidders draw their values independently is readily established by application of Theorem 1 from Athey

(2001) since the case of independent locals ($\gamma = 0$) is the only instance of the model when the *Single Crossing Condition for games of incomplete information (SCC)* is satisfied. However, this theorem cannot be applied directly since it does not guarantee existence of the symmetric equilibrium for symmetric players. In order to tackle this challenge, a modified game where one of the locals is replaced with a player with the same strategy set but with a different objective has to be considered. The modified game satisfies all assumptions from Athey (2001) required to apply existence theorem for games of incomplete information. The equilibrium strategies of the modified game form a symmetric equilibrium of the actual game. See appendix for the complete proof. □

Rough intuition behind the single crossing condition is as follows: whenever all opponents of a player use nondecreasing strategies (in the sense that higher types select higher actions), the player's best response strategy is also nondecreasing.

Clearly, when the correlation between local bidders' values is positive, the SCC is not satisfied. A nondecreasing strategy of the local bidder 1 implies a higher probability of winning for the local bidder 2. As a result, the local bidder 2, when she gets a higher value, might prefer to bid lower since the local bidder 1 is already likely to bid higher. Therefore, the single crossing condition is an exceptionally strong property for the environments such as the one considered here.

Fortunately, as shown in the next section, the existence of a Bayesian-Nash equilibrium in monotone strategies when local bidders' values are perfectly correlated can be established using other methods developed in the literature on existence of the equilibrium in the asymmetric first-price auctions for a single-item.

3.1.2 Perfectly Correlated Values ($\gamma = 1$)

Chernomaz and Levin (2010) discussed existence of the equilibrium for the model of the first-price package auction where local bidders always have the same value that corresponds to the case of perfect correlation in our model.

In order to prove the existence of the equilibrium in the case of perfect correlation, we consider the continuous bidding regime and also assume that

F and G are differentiable over $(0, \bar{v}]$ and $(0, \bar{u}]$ respectively, and that their derivatives , f and g are locally bounded away from zero on these intervals.

Some additional notation is convenient. A per-unit value of the global bidder when she acquires both items is denoted s, i.e., $s = u/2$. Note that v and s are distributed on the same interval $[0, \bar{v}]$. Then, the distribution of s is described by a CDF $\widehat{G}(s) = G(2s)$ with density $\widehat{g}(s) = 2g(2s)$. The strategy of the global bidder is $\widehat{B}(s)$ - a per-unit bid giving her per-unit value s. The actual package bid is then $B(u) = 2\widehat{B}(u/2) = 2\widehat{B}(s)$.

Lemma 3 states that a pair of bidding functions that satisfy first-order conditions do form a Bayesian-Nash equilibrium for the first-price package auction.

Lemma 3. *When local bidders' values are perfectly correlated ($\gamma = 1$), a pair of strictly increasing bidding functions $\beta(v)$ and $\widehat{B}(s)$ forms a Bayesian-Nash equilibrium of the first-price package auction if there exists $\bar{b} \in (0, \bar{v})$ such that the inverses $\alpha = \beta^{-1}, A = \widehat{B}^{-1}$ form a solution of the following system of differential equations over $(0, \bar{b}]$:*

$$\begin{aligned} &\tfrac{d}{db}\alpha(b) = \tfrac{F(\alpha(b))}{(A(b)-b)f(\alpha(b))} \qquad && \tfrac{d}{db}A(b) = \tfrac{2\widehat{G}(A(b))}{(\alpha(b)-b)\widehat{g}(A(b))} \\ &\alpha(\bar{b}) = \bar{v} \quad \alpha(0) = 0 && A(\bar{b}) = \bar{v} \quad A(0) = 0 \end{aligned} \tag{3.7}$$

Proof. See Appendix. □

Proposition 2 (Continuous Bidding)**.** *There exists a symmetric (across local bidders) Bayesian-Nash equilibrium of the first-price package auction in pure strictly increasing bidding strategies when values of the local bidders are perfectly correlated, i.e., $\gamma = 1$.*

Proof. See Appendix. □

The central idea of the proof is straightforward. First, observe that the system of differential equations (3.7) also defines an equilibrium of the certain single-item asymmetric first-price auction with two bidders. Then, using the existence results from the extensive literature on existence in the first-price auctions, the existence of the solution to the system (3.7) can be established and, by Lemma 3, this solution is an equilibrium of the first-price package auction.

3.1.3 Positively Correlated Values ($\gamma \in [0, 1]$)

Unfortunately, the methods developed in the equilibrium existence literature can not be applied in general case. However, the existence on both extremes and visual continuity of the approximated bidding functions strongly suggest that the equilibrium exists for any level of correlation between local bidders' values.

In general cases, we provide some equilibrium characterizations for continuous bidding regime assuming that the symmetric equilibrium exists and that the equilibrium bidding functions $B(u)$ and $\beta(v)$ satisfy the following conditions:

1. $B(u) = \underline{B} = 0 \quad \forall u \in [0, \hat{u}) \quad 0 \leq \hat{u} < \bar{u}$
2. $\beta(v) = \underline{b} = 0 \quad \forall v \in [0, \hat{v}) \quad 0 \leq \hat{v} < \bar{v}$
3. $B(u)$ is strictly increasing on $[\hat{u}, \bar{u}]$
4. $\beta(v)$ is strictly increasing on $[\hat{v}, \bar{v}]$

Conditions 1 - 4 state that equilibrium bidding functions have to be strictly increasing except for maybe having flat segments starting at the lowest value. Lemma 4 shows that in any equilibrium that satisfies these properties, the global bidder always has a strictly increasing strategy while the local bidders equilibrium strategy always includes a non-trivial flat segment unless local bidders' values are perfectly correlated ($\gamma = 1$).

Lemma 4 (Continuous Bidding)**.** *In any equilibrium satisfying conditions 1-4, the following properties hold:*

1. *$B(u)$ is strictly increasing on $[0, \bar{u}]$ (i.e. $\hat{u} = 0$).*
2. *If $\gamma = 1$ then $\beta(v)$ is strictly increasing on $[0, \bar{v}]$ (i.e. $\hat{v} = 0$).*
3. *If $\gamma < 1$ then $\beta(v) = 0$ on $[0, \hat{v}]$ where $\hat{v} > 0$.*

Proof. See Appendix. □

The next section provides the existence result for the first-price auction without package bids.

3.2 Equilibrium Analysis of the First-Price Auction without Package Bids

When package bids are not available, the global bidder has to compete for items separately in two simultaneous first-price auctions using two separate bids, b_1^g and b_2^g. However, Lemma 5 shows that as long as the local bidders follow the same bidding strategy the global bidder is always better off by submitting exactly the same bid in both auctions. Intuitively, such bidding strategy reduces her exposure risk by reducing probability of winning just one item but not both of them.

Lemma 5. *If local bidders follow the same nondecreasing strategy $\beta_l(v)$, the global bidder prefers to submit the same bid in both auctions, i.e., $b_1^g = b_2^g$.*

Proof. See Appendix. □

Similar to the first-price package auction considered in the previous section, we are unaware of any results on the existence of the equilibrium in such an environment. The model is complicated by the possibility of the ex post negative payoff of the global bidder. This is a very distinctive feature since in the model where complementarities are not extreme, like in Chernomaz and Levin (2010), the global bidder can guarantee nonnegative ex-post payoff similar to the standard setup of the first-price auctions.

However, it is straightforward to establish existence of the Bayesian-Nash equilibrium in this model for the discrete bidding regime. Unlike the first-price package auction, a positive correlation between local bidders' values does not lead to a failure of the single-crossing condition in this game since local bidders have to win their items independently from each other. Therefore, monotonicity, characterization and existence results can be easily established for any correlation between local bidders' values ($\forall\gamma \in [0,1]$). The proofs are omitted since they are virtually the same as in section 3.1.1 where the single-crossing condition for games of incomplete information holds because local bidders' values are independent ($\gamma = 0$).

Without loss of generality, local bidders choose their actions from a finite set $S^l = \{b^0, b^1, ..., b^{k_l}\}$ and the global bidder selects her bid from a finite set $S^g = \{B^0, B^1, ..., B^{k_g}\}$ where $k_g = \{j \in \mathbb{N} : B^j \le \bar{v},\ B^{j+1} > \bar{v}\}$. By Lemma 5, in a symmetric equilibrium the global bidder submits the same bid in both auctions. Her equilibrium bid function is denoted $\beta_g(u)$ where u is the value she obtains if she wins both items.

Lemma 6. *In a symmetric equilibrium, both bidding functions $\beta_l(v)$ and $\beta_g(u)$ are nondecreasing.*

Proof. Similar to the proof of Lemma 1. □

Lemma 7. *A pure-strategy Bayesian-Nash symmetric equilibrium is characterized by a pair of step-functions with the following functional forms:*

$$\beta_l(v) = \begin{cases} b^i & \text{if } v \in [s_i, s_{i+1}) \quad 0 \le i \le r \\ b^r & \text{if } v = s_{r+1} \end{cases}$$

and

$$\beta_g(u) = \begin{cases} B^j & \text{if } u \in [t_j, t_{j+1}) \quad 0 \le j \le q \\ B^q & \text{if } u = t_{q+1} \end{cases}$$

where

1. $0 = s_0 < s_1 \le ... \le s_{r+1} = \bar{v}$
   $0 = t_0 < t_1 \le ... \le t_{q+1} = \bar{u}$

2. $0 \le r \le \min[k_l, r^*(q)]$ *where* $r^*(q) = \{i \in \mathbb{N} : b^{i-1} \le B^q,\ b^i > B^q\}$
   $0 \le q \le \min[k_g, q^*(r)]$ *where* $q^*(r) = \{j \in \mathbb{N} : B^{j-1} \le b^r,\ B^j > b^r\}$

Proof. Similar to the proof of Lemma 2. □

Proposition 3 (Discrete Bidding)**.** *There exists a symmetric (across local bidders) Bayesian-Nash equilibrium of the first-price auction without package bids in pure nondecreasing strategies.*

Proof. Similar to the proof of Proposition 1. □

In the next section we describe the numerical approximation technique which can be effectively used to solve both versions of the first-price auction.

4 Numerical Approach

There are a lot of numerical methods suggested in the literature for solving first-price auctions for a single-item. A pioneering contribution was made in Marshall et al. (1994), and further, this topic was expanded by Riley and Li (1997) and Bajari (2001)[13].

One way to compute equilibrium bidding functions is a simple best-response iteration technique. The method provides a certain degree of robustness but tends to be very slow and the convergence is not guaranteed. Other methods approximate equilibrium bidding functions by assigning them some flexible parametric functional forms such as low-order polynomials or piece-wise linear functions and solving the first-order conditions. They are often found to produce highly accurate approximations for the unknown bidding functions in view of their typical smoothness and can be reasonably fast, especially with a good starting guess.

One of the most effective ways to solve the asymmetric first-price auction for a single-item is the backward shooting algorithm, which does not rely on any functional form assumptions. The only disadvantage of the backward shooting routines in a single-item environment is the need for the explicit search for the starting value (the maximum bid), which often results in a slow convergence.

However, in the package environment, any effective use of the shooting methods is highly unlikely. Consider our model of the first-price package auction. The system of equations, which defines a pair of unknown bidding functions, is no longer formed by ordinary differential equations[14], which is the crucial part of the backward shooting algorithms. The key idea behind any shooting routine is the possibility to recover unknown bidding functions from the system of equations in a step-by-step manner relying exclusively on the information received at the previous steps of the routine. In contrast, the system of equations for the first-price package auction modeled in the paper necessarily includes integral terms that represent the two-way nature of the optimal bidding decision on the local side of the market. Thus, any shooting-type routine requires an initial guess for the unknown bidding functions as well

[13] See also Armantier and Richard (1997) and Gayle and Richard (2008).

[14] According to Lemma 4, the system can be represented as a system of ODEs in case of perfect correlation ($\gamma = 1$)

as an explicit search for several variables ($\bar{b}, \bar{B}$ and $\hat{v}$). While the explicit search only affects the computational speed, the need for the initial guess of the unknown bidding functions makes the shooting algorithms completely impractical even in the simple package environments such as ones studied in this paper.

We suggest a new numerical technique that makes use of the discrete formulation of the model[15]. According to lemmas 2 and 7, any symmetric equilibrium bid functions for local bidders and the global bidder are just step functions that are fully characterized by two sets of "jump" points (values at which a bidder prefers to switch from one bidding level to a higher bidding level), $\mathbf{s} = (s_0, s_1, ..., s_r, s_{r+1})$ and $\mathbf{t} = (t_0, t_1, ..., t_q, t_{q+1})$.

The main challenge associated with the system of equilibrium equations and inequalities is the possibility that some bid levels are not played in equilibrium. For example, some bidding levels ($b^i > b^r$ for local bidders and $B^j > B^q$ for the global bidder) are not part of the equilibrium just because they are too high. The maximum equilibrium bid levels (b^r and B^q) have to be determined simultaneously with calculation of vectors $\mathbf{s}$ and $\mathbf{t}$. Another complication might arise when a bidder skips a bidding level (or several bidding levels) that is smaller than the maximum bid level ($b^i < b^r$ for local bidders and $B^j < B^q$ for the global bidder). Such jumps are difficult to handle since there is no way to know which levels are skipped in the equilibrium. We are going to ignore such possibility and look for an equilibrium where all bidding levels up to b^r and B^q are played with positive probability, i.e., $s_i < s_{i+1}$ for $0 \leq i \leq r$ and $t_j < t_{j+1}$ for $0 \leq j \leq q$. While this assumption is restrictive, the typical bidding functions in first-price auctions do satisfy it.

The usual equilibrium system for the local bidders consists of r equations, which determine a vector of "jump" points $\mathbf{s}$, and one inequality, which ensures that the maximum-type local bidder does not profit from bidding b^{r+1}.

$$\begin{cases} \pi_L(s_i, b^{i-1}) & = & \pi_L(s_i, b^i) & 1 \leq i \leq r \\ \pi_L(\bar{v}, b^r) & \geq & \pi_L(\bar{v}, b^{r+1}) & \end{cases} \tag{4.1}$$

[15] The best-response iteration method based on a discrete bidding regime for solving asymmetric first-price auctions for a single-item was suggested in Athey (1997) and used in Athey, Coey and Levin (2010).

One way to solve for the equilibrium of the system (4.1) is the "guess and verify" method used in the backward-shooting algorithms where the initial guess for maximum bid is updated until convergence. In the discrete version, it is equivalent to an explicit search for b^r. The novelty of our numerical technique is the way to endogenize the search for the maximum bids b^r and B^q by using a complementarity formulation instead of the standard one. There are numerical methods specifically designed to solve complementarity problems[16].

The complementarity formulation of the equilibrium system for the local bidders consists of $2k_l$ inequalities that complement each other pairwise. The bid level b^{k_l} is the maximum bid level the local bidders can use in any equilibrium.

$$\left\{\begin{array}{c} \pi(s_i, b^{i-1}) - \pi(s_i, b^i) \geq 0 \\ \textit{complements} \\ \bar{v} \geq s_i \end{array}\right. \qquad 1 \leq i \leq k_l$$

$$\text{or} \tag{4.2}$$

$$\left\{\begin{array}{rcl} \pi(s_i, b^{i-1}) - \pi(s_i, b^i) & \geq & 0 \\ \bar{v} - s_i & \geq & 0 \\ \left[\pi(s_i, b^{i-1}) - \pi(s_i, b^i)\right][\bar{v} - s_i] & = & 0 \end{array}\right. \qquad 1 \leq i \leq k_l$$

Note that the system (4.2) does not depend on r, but instead depends on k_l which is given. Thus, the whole system for both types of bidders can be solved directly without guessing maximum bids b^r and B^q.

Examples of the equilibrium bidding functions in the package and non-package first-price auctions when bidders' values are uniformly distributed ($F(v) = v$ on $[0, 1]$ and $G(u) = u/2$ on $[0, 2]$) are provided in Figure 1 and Figure 2. Both bidding grids Δ_L and Δ_G are uniform with 200 bidding levels on $[0, 1]$ interval. While approximated bidding functions are step-functions, the examples, as shown, are smoothed for a better exposition.

As can be seen from Figure 1, all bidding functions are in perfect accord with the Lemma 4 which describes the shapes and patterns of the equilibrium bid

[16] For example, the PATH solver developed by Steven Dirkse, Michael Ferris and Todd Munson and the FilterMPEC solver by Roger Fletcher and Sven Leyffer

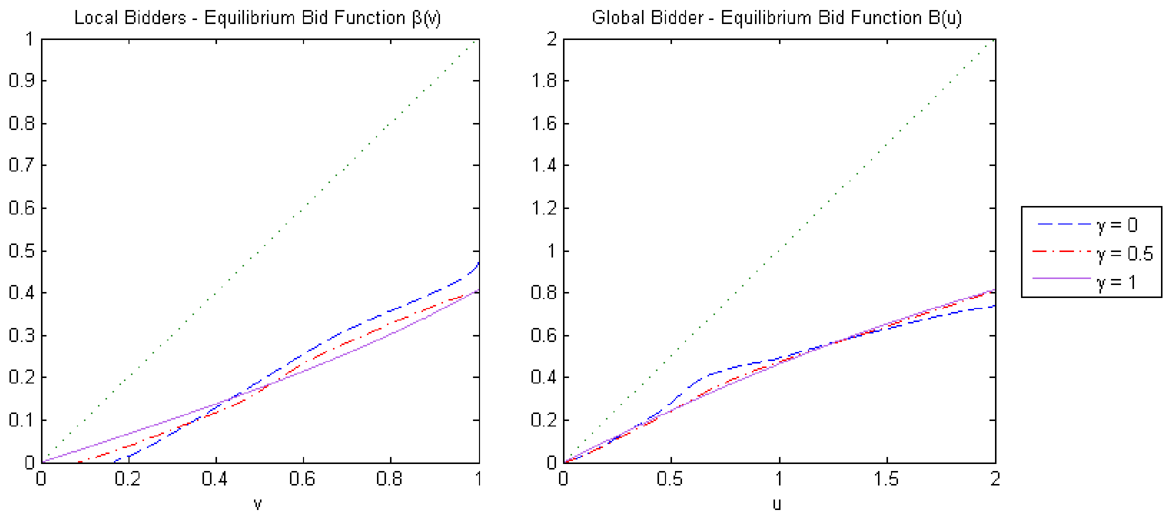

Figure 1: Equilibrium Bids: First-Price Auction with Package Bids

functions in the first-price package auction. When local bidders' values are independent, a local bidder bids zero when her type is sufficiently low expecting a high enough bid from the other local bidder. With an increase in correlation, the size of the zero-bid interval decreases. Intuitively, when correlation is high, a local bidder with low value no longer expects a sufficiently high bid from the other local bidder since with high probability the other local bidder has the same low value. Also, observe that that the maximum total bid from local bidders is around 0.9 while the global bidder maximum bid is around 0.7 when local bidders' values are independent ($\gamma = 0$). However, as correlation between local bidders' values goes up ($\gamma \uparrow$), the distance between maximum bids diminishes quickly.

Figure 2 demonstrates equilibrium bid functions for the first-price auction without package bids. When package bids are not allowed, the global bidder has to compete for items separately and face the exposure risk. When local bidders' values are independent, the global bidder underbids when her value is low and bids more aggressively when her value is high in an attempt to avoid winning just one out of two items. However, with substantial positive correlation between local bidders' values, the impact of the exposure problem is limited. For example, when values of the local bidders are perfectly correlated, the global bidder is never exposed and so she bids accordingly.

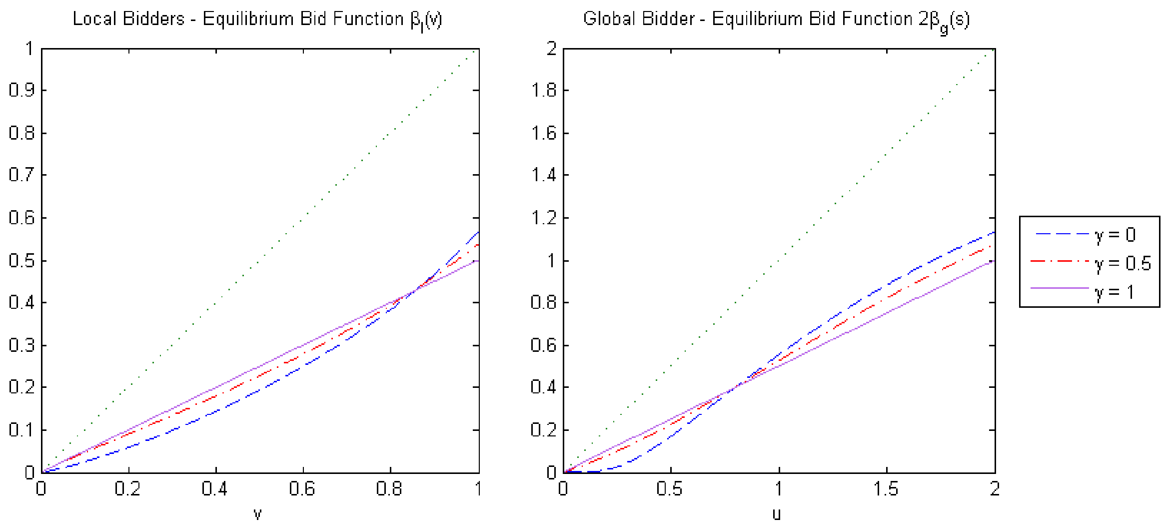

Figure 2: Equilibrium Bids: First-Price Auction without Package Bids

The quantitative analysis of the free-rider and exposure problems is presented in the next section.

5 Free-Rider Problem versus Exposure Problem

This section presents several illustrative examples that show the mechanics of the trade-off between the free-rider and exposure problems. The numerical technique developed in the previous section is used to approximate equilibrium bidding functions in all considered examples. Various relevant auction characteristics, such as revenue and efficiency, are calculated using simulations.

In addition to the two first-price auctions discussed in the paper, we also consider a first-price auction where both items are sold together as one lot and both local bidders are replaced with one bidder whose value for items is exactly the sum of the local bidders' values, v_1+v_2. Formally, with probability γ, this bidder values both items at $2v$ where v is drawn from $F(v)$ and with a probability $1-\gamma$, her value is v_1+v_2 where v_1, v_2 are drawn from $F(v)$ independently from each other. This auction is a standard first-price auction for a single object. It provides a convenient benchmark for evaluating the impacts of the exposure and free-rider problems since it is completely immune to both of them.

The following distributions are assumed for all examples of this section. The global bidder's value is uniformly distributed on [0,2]. The underlying distribution function for local bidders' values is $F(v) = v^{\alpha},\ \alpha > 0$ on $[0, 1]$. When $\alpha = 1$ (uniform distribution), the value distributions are symmetric in a sense that for any γ, the value distribution of the global bidder is the mean-preserving spread of the total value distribution of local bidders. For example, when $\gamma = 0$, the total value $v_1 + v_2$ is distributed according to the triangular distribution on $[0, 2]$, and if local bidders' values are perfectly correlated ($\gamma = 1$), the total value $v_1 + v_2$ is distributed uniformly on $[0, 2]$.

The parameter α can be interpreted in the following way. When α is less than 1, the sum of the local bidders' values is expected to be small in comparison with the expected value of the global bidder, implying that the local bidders lose more frequently under full efficiency. When α is greater than 1, the situation is reversed, with the global bidder winning less frequently under truthful bidding. In other words, a high α makes the local bidders' distribution more advantageous in comparison with that of the global bidder.

The first example, based on the model with uniform distributions ($\alpha = 1$), demonstrates the mechanics of the efficiency trade-off between the free-rider and exposure problems (the right panel of Figure 3). When γ is high, the global bidder easily avoids any exposure risk by submitting the same bid in both markets since local bidders' bids are likely to be very close to each other. In such environments, the first-price auction without package bids achieves high efficiency similar to the efficiency of the benchmark first-price auction. Meanwhile, the first-price package auction is relatively inefficient because of the local bidders' free-riding incentives. Therefore, package bids can hurt efficiency when the exposure risk faced by bidders with complementarities is relatively small.

However, the bids submitted by local bidders can be extremely unequal when their values are slightly correlated or independent from each other ($\gamma = 0$). If this is the case, the exposure risk of the global bidder is high. In the equilibrium, she adjusts her bidding strategy accordingly, by underbidding when her value is low and overbidding when her value is high (see Figure 2).

Despite these adjustments, she often wins only one item. Such an outcome is the major inefficiency driver in the first-price auction without package bids.[17] Meanwhile, the package auction is completely immune to such outcomes. Changes in γ do affect the equilibrium bid functions, but the overall efficiency of the first-price package auctions stays relatively constant at high levels. Therefore, package bids can significantly improve efficiency performance of the first-price auction, especially in environments with a high exposure risk.

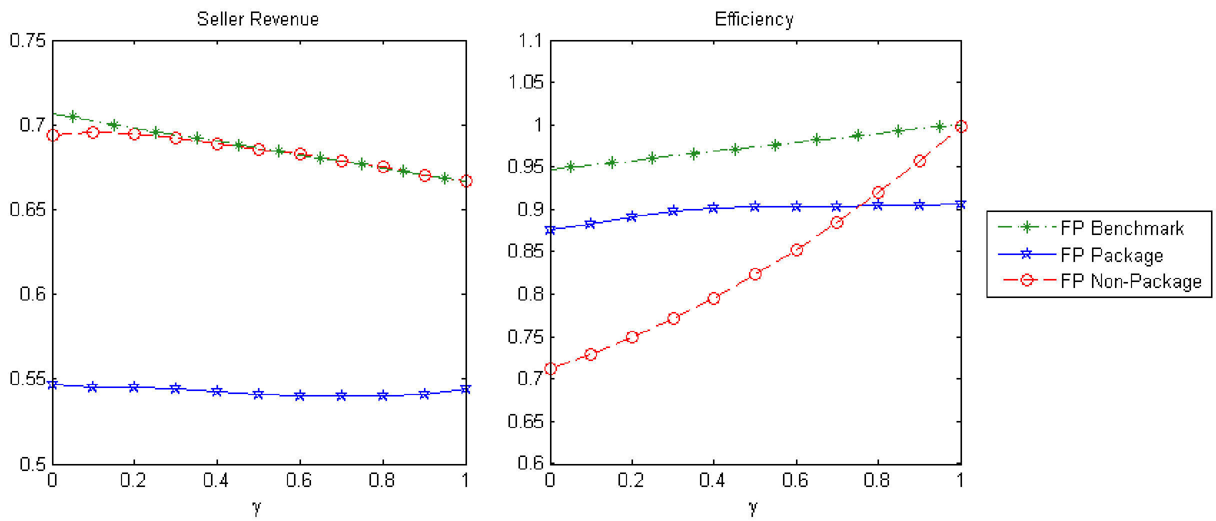

Figure 3: Seller Revenue and Efficiency: $F(v) = v$ $(\alpha = 1)$

A differential bid shading, an inevitable property of all first-price auctions, is responsible for inefficiencies arising from bidder asymmetries. In the standard first-price auctions for a single-item the numerous dimensions of asymmetry are limited to a simple distributional asymmetry. Package bids introduce yet another degree of asymmetry to the first-price auctions by allowing bidders with different value structures to choose package bids that better fit their preferences. Such bid asymmetries can reduce or increase inefficiencies generated by other forms of asymmetries.

The next example demonstrates that the free-rider problem in the first-price

[17]When local bidders' values are independent ($\gamma = 0$), nearly 26% out of 28% of inefficiencies arise from outcomes where the global bidder wins in one auction but loses in the other one.

package auction can actually improve efficiency of the benchmark first-price auction. Consider an environment where the underlying value distribution of the local bidders, say $F(v) = \sqrt{v}$ ($\alpha = 0.5$), is comparatively worse than the uniform value distribution of the global bidder. The efficiency performance of all auctions in this environment can be found in the right panel of Figure 4. Both the benchmark first-price auction and the non-package first-price auction are inefficient because of the distributional asymmetry. Meanwhile, the bidding asymmetry of the package auction induces the free-rider problem that helps mitigate the distributional differences. As a result, the first-price package auction is highly efficient in spite of the distributional asymmetry. In fact, in this particular example it is fully efficient when local bidders' values are perfectly correlated ($\gamma = 1$). Intuitively, the free-rider problem between local bidders results in a significant bid shading on their part. Using terminology from Maskin and Riley (2000), the global bidder is a *weak bidder* who bids more aggressively while both local bidders together represent a *strong bidder* who bids less aggressively.[18] Therefore, a more advantageous value distribution of the global bidder evens out the differential shading incentives by making the bidding of the global bidder less aggressive while also promoting more competitive bidding from the local bidders by reducing their free-rider incentives.

Note that such asymmetric environments might be empirically relevant. For example, in procurement auctions, big (global) suppliers might have a better cost distribution than small (local) suppliers.

In both examples considered above, the first-price package auction generates significantly lower revenue than the other two formats. Intuitively, the free-rider problem negatively affects revenue since all bidders have incentives to bid lower in comparison with the benchmark first-price auction. Local bidders reduce their bids in an attempt to free-ride on each other, while the global bidder submits a lower bid in response. Meanwhile, the exposure problem can potentially lead to either an increase or a decrease in revenue. On the one hand, the global bidder shades her bid when her value is low since the probability that she wins both items is small. On the other hand, she can substantially overbid when her value is high in order to reduce her exposure risk. Figures 3 and 4 provide examples when the non-package first-price

[18] In Maskin and Riley (2000), a strong bidder bids less aggressively because her value distribution first-order stochastically dominates the value distribution of the weak bidder.

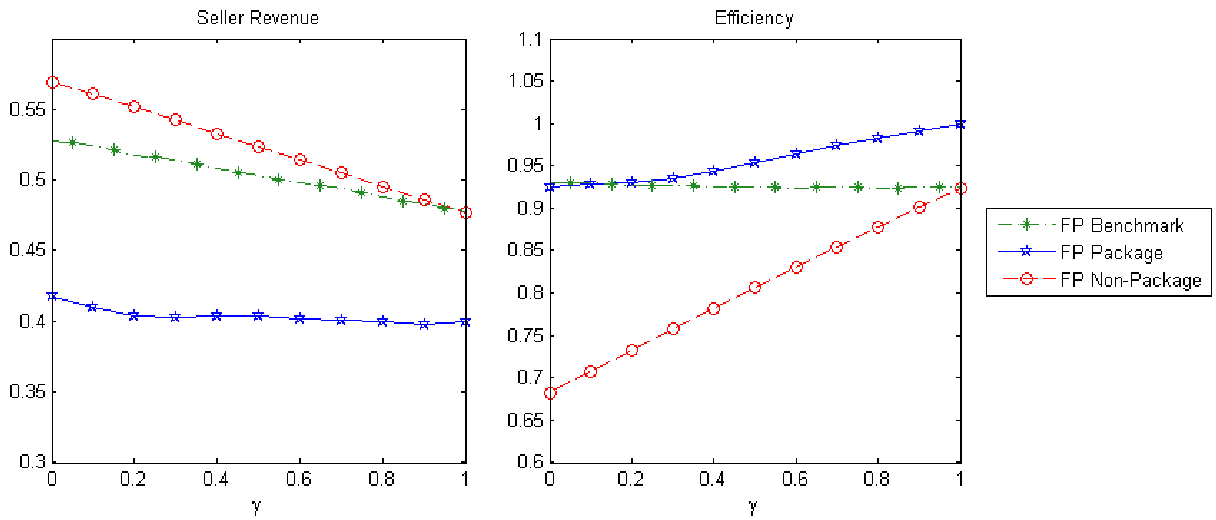

Figure 4: Seller Revenue and Efficiency: $F(v) = \sqrt{v}$ ($\alpha = 0.5$)

auction generates lower and higher revenues than the benchmark first-price auction. However, as the next example shows, the revenue performance of the first-price package auction highly depends on the environment.

Consider the following modification of the original model. It consists of two homogeneous items, one global bidder and three local bidders. With homogeneous items, local bidders do not care which particular items they win. However, all local bidders still have value only for one unit and the global bidder still needs both units. In this setup, the first-price auction without package bids is a simple pay-as-bid auction where bidders submit demand curves.

In such environments, the negative revenue impact of the free-rider problem is relatively small since local bidders have to compete not only with the global bidder but also with each other. This competition among local bidders partially mitigates free-rider incentives and substantially improves revenues.

Under the continuous bidding regime, this model does not have a pure Bayesian-Nash equilibrium if local bidders' values are positively correlated[19]

[19]With probability $\gamma > 0$ a local bidder can perfectly predict the exact bids of the other local bidders and can easily increase her bid by a small amount such that she always gets an item whenever the local bidders win.

in the same way as in the original model of Section 2. Therefore, it is assumed that all values are independent ($\gamma = 0$). For the distribution functions used in the previous example, the first-price package auction achieves almost 4% higher revenue and 23% higher efficiency than the first-price auction without package bids.

To sum up, the first-price package auction in environments with sufficient competition among free-riding bidders seems to result in revenue and efficiency improvements. Thus, the frequent use of the first-price package design in practical applications can be explained not only by its simple description but also by its performance characteristics.

In the next section we compare the first-price package auction with other package alternatives suggested in the literature.

6 Core-Selecting Auctions

Core-selecting auctions have recently been suggested as alternatives to the Vickrey-Clarke-Groves (VCG) mechanism. While the mechanism has the attractive property that truth-telling is a dominant strategy — and truth-telling by all participants in the VCG mechanism implies full efficiency — there are several problems with the VCG in environments with complementarities. This is ironic since exactly these environments are used to motivate the use of package bids. The list of problems includes a possibility of extremely low revenue for the seller (sometimes even zero revenue!), a non-monotonicity of the seller revenue with respect to the bidders' values and high vulnerability to exotic bidders' strategies such as shill-bidding. The main reason for all mentioned disadvantages of the VCG mechanism in environments with complementarities is that sometimes its payment vector falls outside the core in a sense that there exists a coalition of bidders who cumulatively offered to pay more for the same subset of items.

Such practical flaws of the VCG mechanism have trigged both theoretical and applied interests in alternative mechanisms that came to be known as *core-selecting auctions*. The main ingredient of the novel auction design is the *core property*, which guarantees that the payments collected by the auctioneer are always "sufficiently high." In general, payments greater than those of the VCG mechanism are unable to support truthful bidding as the dominant strategy equilibrium and potentially may lead to substantial underbidding. In order to *minimize* bidders' incentives to deviate from truthful bidding, Day and Milgrom (2008) suggested to minimize seller's revenue subject to the core constraints. Core-selecting auctions that minimize the seller's revenue are known as *minimum-revenue core* auctions.[20] The total payment to the seller in such auctions necessarily coincides with that of the VCG when the VCG payment vector belongs to the core[21] and is strictly greater when it lies outside the core.

As has been noted above, even the minimum-revenue core auctions, in general, cannot induce truth-telling incentives to all bidders in environments with complementarities. Consider our model where the global bidder submits a package bid B for both items and local bidders bid b_1 and b_2 on individual items. In minimum-revenue core auctions, local bidders have to pay the global bidder's bid B whenever they win. However, the exact split of the total payment B between local bidders necessarily depends on their individual bids b_1 and b_2. As a result, local bidders face the free-rider problem, which is similar to the one they have in the first-price package auction.

Intuitively, the free-rider problem in first-price package auctions is worse. First, bidders have stronger incentives to free-ride on each other since any decrease in the bidder's bid is matched one-to-one by a decrease in the bidder's payment if she wins. Meanwhile, payments in minimum-revenue core auctions are either unaffected or decrease partially (50 cents per a $1 drop) in response to bid reductions. Second, the free-rider problem in the first-price package auction triggers optimal response by other bidders who try to take advantage of their opponents' low bids. In the context of our model, the global bidder bids less competitively because of the free-rider problem between local bidders. In contrast, in minimum-revenue core auctions the global bidder incentives are not distorted.[22] However, the stronger free-rider

[20] Sometimes minimum-revenue core auctions are referred to as core-selecting auctions.

[21] In environments without complementarities, the VCG payment vector is always in the core. For example, consider a simple single-item private-value environment with several bidders. In this setup, a core-selecting auction is an auction that allocates the item to the highest bidder and requires the winner to pay an amount between her bid and the second-highest bid. Also note that the minimum-revenue core auctions, VCG mechanism and the second-price auction are all equivalent in this environment.

[22] Truth-telling is a weakly dominant strategy for the global bidder in minimum-revenue core auctions.

problem of the first-price package auction does not necessarily imply lower revenue because of the usual difference between first-price and second-price auctions. The global bidder is more likely to win in "second-price-like" minimum-revenue core auctions and pay very little because her payment is a sum of local bidders' bids affected by their free-rider problem. Therefore, revenue generated by minimum-revenue auctions can be extremely low (sometimes even zero revenue) while revenues in the first-price package auctions are always positive.

Several core-selecting auction rules have been analyzed by Ausubel and Baranov (2010) in an incomplete information environment using the same model with two items and three bidders. Under some distributional assumptions, authors derived closed-form solution for all considered minimum-revenue core auctions including the proxy rule introduced by Ausubel and Milgrom (2002, 2006) and the nearest-Vickrey rule suggested by Day and Raghavan (2007) and Day and Cramton (2009). Using their solutions, we can compare the revenue and efficiency performance of the first-price package auction with those of the leading minimum-revenue core designs. The results are summarized in Figure 5.

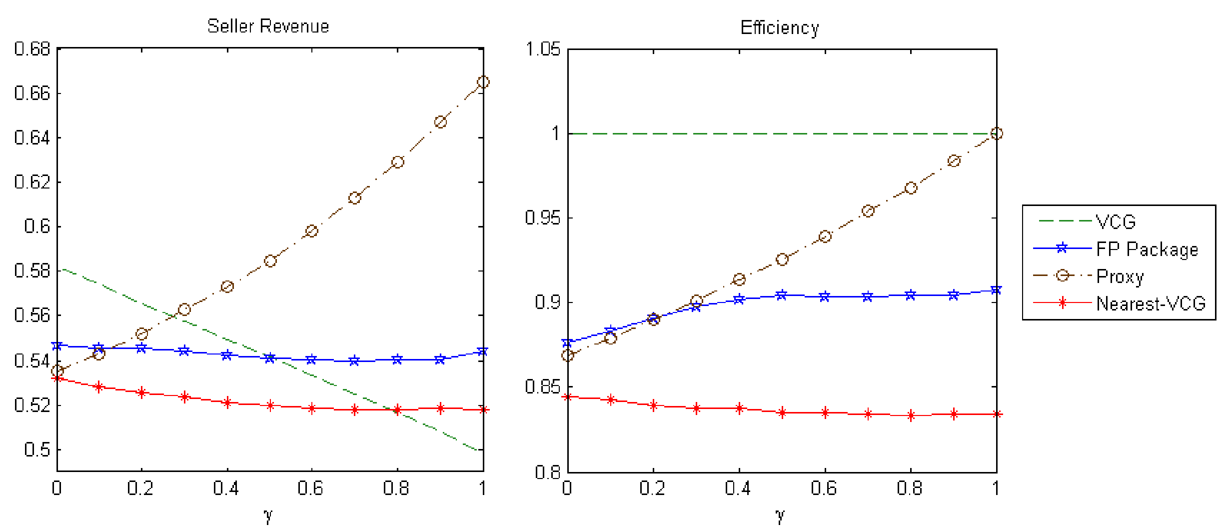

Figure 5: Seller Revenue and Efficiency: $F(v) = v$ $(\alpha = 1)$

The first-price package auction performs reasonably well in comparison with the proxy auction and the nearest-Vickrey auction, in terms of both revenue and efficiency, despite the more serious free-rider problem. For example, the first-price package auction generates higher revenue and efficiency than the nearest-Vickrey auction for any γ. At the same time, the proxy rule generates higher revenue and achieves higher efficiency than the first-price package auction when correlation between local bidders' values is sufficiently high. This is due to the fact that a substantial positive correlation effectively mitigates the free-rider problem in the proxy auction. Corresponding numbers for revenue, efficiency and profits of bidders for all auction rules can be found in Table 1.

γ	Statistics	VCG	Proxy*	N-VCG*	FP Pack	FP Non-Pack
$\gamma = 0$	*Revenue*	0.5833	0.5360	0.5327	0.5471	0.6940
	Efficiency	1	0.8679	0.8431	0.8762	0.7126
	Profit Global	0.2916	0.4642	0.4673	0.4269	0.2027
	Profit Local	0.2087	0.1342	0.1335	0.1498	0.1501
$\gamma = 0.5$	*Revenue*	0.5417	0.5852	0.52	0.5412	0.6857
	Efficiency	1	0.9261	0.8356	0.9039	0.8236
	Profit Global	0.3126	0.4148	0.4798	0.4304	0.2604
	Profit Local	0.2295	0.1523	0.1415	0.1647	0.1555
$\gamma = 1$	*Revenue*	0.5	0.6667	0.5185	0.5445	0.6666
	Efficiency	1	1	0.8334	0.9073	1
	Profit Global	0.3335	0.3335	0.4816	0.4270	0.3324
	Profit Local	0.2499	0.1666	0.1481	0.1754	0.1665

* - Based on Ausubel and Baranov (2010)

Table 1: Revenue, Efficiency and Profits

7 Conclusion

This paper contributes to the quickly expanding literature on the use of combinatorial bids in multi-object auctions. In environments with complementarities, non-package auction designs can easily fail to achieve efficient allocations and generate low revenues for the seller because they do not allow bidders to express their synergies across items. At the same time, package auctions easily handle complementarities of any complexity, but instead introduce free-riding incentives that can decrease or completely mitigate any gains attained from exploiting synergies. Moreover, combinatorial auctions are also more complex technically and computationally, mainly because of the large number of possible packages even for a small number of items. Therefore, a careful cost-benefit analysis of package designs is an important market design question.

The impact of package bids on first-price auctions is of substantial interest since they are the most frequently used package auctions in applications, especially in the public procurement. Unfortunately, an inherent asymmetry among bidders' bids and values, required for a non-trivial comparison between package and non-package designs, complicates the analysis of first-price auctions which are proved to be exceptionally tedious in asymmetric environments.

In the simple model with two types of bidders, we demonstrate the impact of the package bids on the first-price auction. The model, while simple and intuitive, includes a number of realistic features that motivate the use of package auctions, such as the presence of substantial complementarities in bidders' preferences and a positive correlation of bidders' values.

We perform a Bayesian-Nash equilibrium analysis of the first-price auction with and without package bids. For the package auction, we prove the existence of the equilibrium in the case when all bidders' values are independent and also when some bidders' values are perfectly correlated. For the non-package auction, we provide a general equilibrium existence result for our model. A discrete bidding regime, i.e., all bids are restricted to some discrete bidding grids, is used in the majority of the proofs.

We also develop a novel numerical technique, based on the discrete bidding regime, which can be effectively used to approximate equilibrium bidding strategies in first-price auctions. The key element of this procedure is the complementarity formulation for the system of equilibrium inequalities. Our experience suggests that this technique can also be successfully applied in other areas. For example, it is a well-behaved and exceptionally fast alternative for a lot of different numerical methods developed for approximating unknown bidding strategies in asymmetric single-item first-price auctions.

Armed with the numerical method, we take a close look at the forces behind the exposure and free-rider problems in the first-price auctions. Using several examples, we demonstrate the exact mechanics of the efficiency trade-off between package and non-package designs. Even in the environments with extreme forms of complementarities, package bids can be very harmful in terms of revenue and efficiency when the exposure risk faced by the bidders with complementarities is rather small. However, when bidders face a high probability of exposure, package bids can dramatically improve efficiency.

Moreover, a flexibility of package bids introduces bid asymmetries in the first-price auction. Such bid asymmetries can also increase efficiency of the package auction by reducing inefficiency of the non-package design related to the distributional asymmetries. For example, the first-price package auction is highly efficient when bidders with large demands have a more advantageous per-unit distribution than the bidders with small demands.

Finally, we show that in environments that are more competitive than the one considered in the paper, the first-price package auction can be superior to the first-price non-package auction in both revenue and efficiency. We also compare the first-price package auction with the leading package alternatives such as the Vickrey-Clarke-Groves mechanism and core-selecting auctions. In the environment considered, the first-price package auction demonstrates very strong performance characteristics.

These findings suggest that in multi-object environments with complementarities, package designs, and the first-price package auction in particular, indeed can deliver a better performance in comparison with their non-package alternatives.

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8 Appendix

8.1 Proof of Lemma 1

We prove that $\beta(v)$ is nondecreasing. The proof for $B(u)$ is similar. First, observe that the probabilities of winning with the lowest possible bids (b^0 or B^0) are strictly positive for all bidders in any equilibrium, i.e., $Pr_L^0 > 0$ and $Pr_G^0 > 0$. Therefore, in equilibrium all probabilities of winning are strictly positive according to (3.4).

Now suppose that $\beta(v) = b^i$. Then, using IC constraints (3.1), for all $k \in \mathbb{N}$ we get:

$$\pi_L(v, b^i) \geq \pi_L(v, b^k)$$

$$(v - b^i)Pr_L^i \geq (v - b^k)Pr_L^k$$

$$v(Pr_L^i - Pr_L^k) \geq b^i Pr_L^i - b^k Pr_L^k$$

Note that above inequalities still hold if v is replaced with $v' > v$ as long as $Pr_L^i \geq Pr_L^k$. Thus, $\forall k \in \mathbb{N} : k \leq i$

$$v'(Pr_L^i - Pr_L^k) \geq b^i Pr_L^i - b^k Pr_L^k$$

$$(v' - b^i)Pr_L^i \geq (v' - b^k)Pr_L^k$$

$$\pi_L(v', b^i) \geq \pi_L(v', b^k)$$

It is easy to argue that a stronger version of the last inequality holds, i.e.:

$$\pi_L(v', b^i) > \pi_L(v', b^k) \quad \forall k \in \mathbb{N} : k < i$$

Suppose that expected profits from playing b^i and $b^k : b^k < b^i$ at v' are the same, but then $\pi_L(v, b^i) < \pi_L(v, b^k)$ since $Pr_L^i > Pr_L^k > 0$ which contradicts our initial assumption that $\beta(v) = b^i$. Consequently, $\beta(v') \geq b^i = \beta(v) \quad \forall v' \geq v$.

□

8.2 Proof of Proposition 1

Consider a modified model with the following payoffs where the global bidder and one of the local bidders have exactly the same payoff while the other local has a different payoff function.

$$
\begin{aligned}
&\pi_G(B,u) = (u-B)Pr(\beta^1(v_1)+\beta^2(v_2) < B) && B \in S_G \\
&\pi_L^1(b_1,v_1) = (v_1-b_1)Pr(b_1+\beta^2(v_2) > B(u)) && b_1 \in S_L \\
&\pi_L^2(b_2,v_2) = -(\beta^1(v_2)-b_2)^2 && b_2 \in S_L
\end{aligned} \tag{8.1}
$$

The idea behind the last payoff in (8.1) is that the local bidder 2 just tries to match the strategy of the local bidder 1 at her value. Thus, in any equilibrium of the modified game: $\beta^1(v) = \beta^2(v) \quad \forall v \in [0, \bar{v}]$ by construction. By Theorem 1 from Athey (2001), the modified game has an equilibrium since all conditions required by this theorem, including SCC, are satisfied. Any equilibrium of the modified game is an equilibrium of the original game since in the original model the local bidders are symmetric. Therefore, there exists a symmetric equilibrium of the original game.

□

8.3 Proof of Lemma 3

A local bidder solves the following optimization problem:

$$
\begin{aligned}
\pi_L(v,b) &= (v-b)Pr(b+\beta(v) \geq 2\widehat{B}(s)) = \\
&= (v-b)Pr\left(s \leq A\left(\tfrac{b+\beta(v)}{2}\right)\right) = \\
&= (v-b)\widehat{G}\left(A\left(\tfrac{b+\beta(v)}{2}\right)\right)
\end{aligned}
$$

F.O.C.:

$$
dA(b) = \frac{2\widehat{G}(A(b))}{(\alpha(b)-b)\widehat{g}(A(b))} \qquad A(\bar{b}) = \bar{v} \qquad A(\underline{b}) = 0 \tag{8.2}
$$

The global bidder faces the following optimization problem:

$$\begin{aligned} \pi_G(s,b) &= (2s-2b)Pr(2b \geq 2\beta(v)) = \\ &= 2(s-b)Pr(v \leq \alpha(b)) = \\ &= 2(s-b)F(\alpha(b)) \end{aligned}$$

F.O.C:

$$d\alpha(b) = \frac{F(\alpha(b))}{(A(b)-b)f(\alpha(b))} \qquad \alpha(\bar{b}) = \bar{v} \qquad \alpha(0) = 0 \tag{8.3}$$

Rewrite (8.2) and (8.3) as:

$$\frac{d}{db}\ln F(\alpha(b)) = \frac{1}{A(b)-b} \qquad \frac{d}{db}\ln \widehat{G}\left(A\left(\frac{b+\beta(v)}{2}\right)\right) = \frac{1}{\alpha(b)-b} \tag{8.4}$$
$$\text{for all} \quad b \in (0, \bar{b}]$$

First, a bid larger than $\bar{b}$ is never a best response for both types of bidders. Second, a bid of 0 is a best response for any bidder with value 0. Third, if bidders value is above 0 biddng above bidder's value is strictly dominated by bidding ,say, $b = (v+0)/2$.

Suppose a local bidder has a value of $v > 0$ and bids $b < v$. Then the logarithm of her positive expected profit ant its derivative are:

$$\begin{aligned} \ln \pi_L(v,b) &= \ln(v-b) + \ln \widehat{G}\left(A\left(\frac{b+\beta(v)}{2}\right)\right) \\ \ln \pi_L(v,b)' &= \frac{-1}{(v-b)} + \frac{1}{\alpha(b)-b} \end{aligned} \tag{8.5}$$

Note that the derivative is strictly negative when $v < \alpha(b)$ (or $\beta(v) < b$) and it is strictly positive when $\alpha(b) < v$ (or $b < \beta(v)$). Therefore, $b = \beta(v)$ is the global maximum.

Same technique can be used to show that bid $b = \widehat{B}(s)$ is the global maximum for the global bidder with per unit value s.

□

8.4 Proof of Proposition 2

Now define $H(x) = \sqrt{\widehat{G}(x)}$ and $h(x) = \frac{\widehat{g}(x)}{2\sqrt{\widehat{G}(x)}}$ and note that:

$$\frac{H(x)}{h(x)} = \frac{2\widehat{G}(x)}{\widehat{g}(x)} \tag{8.6}$$

Plugging (8.6) into (8.2) and (8.3) we get the following system of differential equations:

$$\begin{aligned} dA(b) &= \tfrac{H(A(b))}{(\alpha(b)-b)h(A(b))} \qquad A(\bar{b}) = \bar{v} \qquad A(\underline{b}) = 0 \\ d\alpha(b) &= \tfrac{F(\alpha(b))}{(A(b)-b)f(\alpha(b))} \qquad \alpha(\bar{b}) = \bar{v} \qquad \alpha(0) = 0 \end{aligned} \tag{8.7}$$

Observe that the system of ODEs formed by (8.7) is a standard system for a single item first-price auction with two bidders with their valuations distributed according to cummulative distribution functions $F(.)$ and $H(.)$. Consequently, the existence results from assymetric first-price literature can be applied.

By Theorem 3 from Lebrun (1997), there exists equilibrium of the assymetric first-price auction. This is equivalent to the existence of the solution of the system (8.7) by Theorem 2 from the same paper.

Since G is atomless, so does H. All asummptions of Theorem 3 in Lebrun (1997) are satisfied, and consequently, the system (8.7) has a solution. By Lemma 3, this solution is an equilibrium of the first-price package auction when values of local bidders are perfectly correlated.

□

8.5 Proof of Lemma 4

1. $B(u)$ *is strictly increasing on* $[0, \bar{u}]$ *(i.e.* $\hat{u} = 0$*)*

By characterization assumptions, $B(u)$ is strictly increasing on $[\hat{u}, \bar{u}]$ and constant on $[0, \hat{u})$. The only way $B(u)$ is strictly increasing on the whole interval is when $\hat{u} = 0$. Assume that $\hat{u} > 0$. When $\hat{v} > 0$, a tie at the minimum bid ($\underline{B} = 2\underline{b}$) occur with positive probability which can not be part of equilibrium by standard arguments. When $\hat{v} = 0$ the global bidder with value $0 < u < \hat{u}$ has a profitable deviation since bidding anything above minimum bid and below his value generates a strictly positive expected payoff while bidding minimum bid delivers zero.

2. *If* $\gamma = 1$, $\beta(v)$ *is strictly increasing on* $[0, \bar{v}]$ *(i.e.* $\hat{v} = 0$*)*

Similar to the previous case, I need to show that $\hat{v} = 0$. Assume that $\hat{v} > 0$. A local bidder with value $0 < v < \hat{v}$ has zero probability of winning since the total bid from the local side is $2\underline{b}$. Bidding anything above minimum bid and below her value gives a strictly positive expected payoff.

3. *If* $\gamma < 1$, $\beta(v) = 0$ *on* $[0, \hat{v}]$ *where* $\hat{v} > 0$

Denote $\Phi_i(b_i, v_i)$ and $\phi_i(b_i, v_i)$ the probability of winning and marginal probability of winning for a local bidder who submits a bid assuming all other bidders follow their equilibrium strategies, i.e.:

$$\Phi_i(b_i, v_i) = \gamma \int\limits_{b_i + \beta(v_i) > B(u)} g(u)du + (1 - \gamma) \iint\limits_{b_i + \beta(v_j) > B(u)} f(v_j)g(u)du$$

$$\phi_i(b_i, v_i) = \frac{\partial \Phi_i(b_i, v_i)}{\partial b_i}$$

If $\gamma < 1$, the probability of winning for the local bidder with the lowest bid $\underline{b}$ is greater than zero for all values, i.e. $\Phi(\underline{b}, v) > 0 \quad \forall v \in [0, \bar{v}]$.

Consider a Taylor expansion of $\Phi(\underline{b}, v)$ around $\underline{b}$ for all v:

$$\Phi(b, v) = \Phi(\underline{b}, v) + \phi(\underline{b}, v)(b - \underline{b}) + o((b - \underline{b})^2)$$

Then the problem of the local bidder is as follows:

$$\pi_L(v, b) = (v - b)Pr(b + \beta(v_j) > B(u)) = (v - b)\Phi(b, v)$$

$$\pi_L(v, b) = (v - b) * (\Phi(\underline{b}, v) + \phi(\underline{b}, v)(b - \underline{b}) + o((b - \underline{b})^2))$$

$$b^* \approx \underline{b} + \frac{(v - \underline{b})\phi(\underline{b}, v) - \Phi(\underline{b}, v)}{2\phi(\underline{b}, v)} \tag{8.8}$$

Since $\phi(\underline{b}, v)$ is positive and bounded, there exist v such that $v > 0$ and the second term in (8.8) is negative. Thus, the unconstrained optimal bid b^* is less then $\underline{b} = 0$ since the term $(v - \underline{b})\phi(\underline{b}, v)$ goes to zero when $v \to 0$ while the term $\Phi(\underline{b}, v)$ converges to some positive number.

$\square$

8.6 Proof of Lemma 5

Denote $F_1 = F(\beta_l^{-1}(b_1^g))$ and $F_2 = F(\beta_l^{-1}(b_2^g))$. Without loss of generality, assume that $b_1^g \geq b_2^g$ such that $F_1 \geq F_2$. Further denote p_1 and p_2 expected payments of the global bidder in case she wins either item. The proof is general and does not have to for the first-price auction only. For the first-price auction $p_1 = b_1^g$ and $p_2 = b_2^g$.

Expected profit of the global bidder when she submits bids b_1^g and b_2^g is given by:

$$
\begin{aligned}
\Pi(u, b_1^g, b_2^g) &= \gamma\left[(u - p_1 - p_2)F_2 - p_1(F_1 - F_2)\right] + \\
&\quad +(1-\gamma)\left[(u - p_1 - p_2)F_1F_2 - p_1F_1(1 - F_2) - p_2F_2(1 - F_1)\right] \\
&= \gamma\left[uF_2 - p_1F_1 - p_2F_2\right] + (1-\gamma)\left[uF_1F_2 - p_1F_1 - p_2F_2\right]
\end{aligned}
$$

Then, the following inequality holds:

$$
\begin{aligned}
&\Pi(u, b_1^g, b_1^g) + \Pi(u, b_2^g, b_2^g) - 2\Pi(u, b_1^g, b_2^g) = \\
&\qquad = \gamma\left[u(F_1 - F_2)\right] + (1-\gamma)\left[u(F_1 - F_2)^2\right] \geq 0
\end{aligned}
$$

Or, equivalently:

$$
\Pi(u, b_1^g, b_1^g) \geq \Pi(u, b_1^g, b_2^g) \quad \text{or} \quad \Pi(u, b_2^g, b_2^g) \geq \Pi(u, b_1^g, b_2^g)
$$

Therefore, the expected profit from submitting equal bids are higher then submitting different bids.

$\square$